Continuous Evolution Families

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1. Notation

We denote the complex plane by $\mathbb{C}$. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ be the space of all analytic functions in the unit disk $\mathbb{D}$, endowed with the usual topology of local uniform convergence. Let $\mathfrak{B} := \{f \in \mathcal{A} : f(\mathbb{D}) \subset \mathbb{D}\} \subset \mathcal{A}$. For $\lambda \in \mathbb{D}$, $\sigma_\lambda(z) := \frac{z+\lambda}{1+\bar{\lambda}z}$, $\quad z \in \mathbb{D}$. Let $I \subset [-\infty, \infty]$ be an interval and $I_+^2 := \{(s,t) : s,t \in I \text{ with } s \leq t\}$. $D(I) \subset I_+^2$ is a diagonal set of I_+^2, namely $D(I) := \{(t,t) : t \in I\}$.

2. Definition of Evolution Family

Definition 1. A family $\{\omega_{s,t}\}_{(s,t)\in I_+^2}$ in $\mathfrak{B}$ is said to be an *evolution family* if it has the following three properties;

(EF1) $\omega_{s,t}$ is non-constant for all $(s,t) \in I_+^2$,
(EF2) $\omega_{t,t} = \mathrm{id}_{\mathbb{D}}$ for all $t \in I$,
(EF3) $\omega_{u,t} \circ \omega_{s,u} = \omega_{s,t}$ for $s,u,t \in I$ with $s \leq u \leq t$.

Definition 2. An evolution family $\{\omega_{s,t}\}_{(s,t)\in I_+^2}$ is said to be *jointly continuous* if;

(EF4) the mapping $I_+^2 \ni (s,t) \mapsto \omega_{s,t} \in \mathcal{A}$ is continuous.

Let $I = [a,b]$ with $-\infty < a < b < \infty$.

Definition 3. $\{\omega_{s,t}\}_{(s,t)\in I_+^2}$ is *continuous with respect to the right* (or *left*) *parameter* if the mapping $I \ni t \mapsto \omega_{a,t} \in \mathcal{A}$ (or $I \ni s \mapsto \omega_{s,b} \in \mathcal{A}$, respectively) is continuous.

Definition 4. Let $\{\omega_{s,t}\}_{(s,t)\in I_+^2}$ be an evolution family.

- An evolution family $\{\omega_{s,t}\}_{(s,t)\in I_+^2}$ is *hyperbolically bounded* if
$$\sup_{(s,t)\in I_+^2} |\omega_{s,t}(0)| < 1$$
- An evolution family $\{\omega_{s,t}\}_{(s,t)\in I_+^2}$ is *locally hyperbolically bounded* if $\{\omega_{s,t}\}_{(s,t)\in I_+^2}$ is hyperbolically bounded on any compact subinterval of I.

3. Main theorems

Theorem 5. *Let $\{\omega_{s,t}\}_{(s,t)\in I_+^2}$ be an evolution family on $I = [a,b]$ with $-\infty < a < b < \infty$. Then the following are equivalent;*

(i) *The family $\{\omega_{s,t}\}_{(s,t)\in I_+^2}$ is jointly continuous,*
(ii) *The family $\{\omega_{s,t}\}_{(s,t)\in I_+^2}$ is continuous with respect to the right parameter,*
(iii) *For each fixed $z_0 \in \mathbb{D}$, $\omega_{a,t}(z_0)$ and $\omega'_{a,t}(z_0)$ are continuous functions of $t \in [a,b]$,*
(iv) *For some $z_0 \in \mathbb{D}$, $\omega_{a,t}(z_0)$ and $\omega'_{a,t}(z_0)$ are continuous functions of $t \in [a,b]$.*

Furthermore in these cases $\omega_{s,t}$ is univalent on $\mathbb{D}$ for each $(s,t) \in I_+^2$.

Theorem 6. *Let $\{\omega_{s,t}\}_{(s,t)\in I^2_+}$ be an evolution family on $I = [a,b]$ with $-\infty < a < b < \infty$. Then the following are equivalent;*

(i) *The family $\{\omega_{s,t}\}_{(s,t)\in I^2_+}$ is jointly continuous,*

(v) *The family $\{\omega_{s,t}\}_{(s,t)\in I^2_+}$ is hyperbolically bounded and continuous with respect to the left parameter,*

(vi) *The family $\{\omega_{s,t}\}_{(s,t)\in I^2_+}$ is hyperbolically bounded, $\lim_{\substack{s\le c\le t\\ t-s\searrow 0}} \omega_{s,t}(0) = 0$ for all $c \in [a,b]$ and $\omega'_{s,b}(0)$ is a continuous function of $s \in [a,b]$ with $\omega'_{s,b}(0) \neq 0$ for $s \in [a,b]$.*

Theorem 7. *Let $\{\omega_{s,t}\}_{(s,t)\in I^2_+}$ be an evolution family on $I = [a,b]$ with $-\infty < a < b < \infty$. Then the following are equivalent;*

(i) *The family $\{\omega_{s,t}\}_{(s,t)\in I^2_+}$ is jointly continuous,*

(vii) *The family $\{\omega_{s,t}\}_{(s,t)\in I^2_+}$ is jointly continuous on $D(I)$,*

(viii) *The family $\{\omega_{s,t}\}_{(s,t)\in I^2_+}$ is hyperbolically bounded with $\lim_{\substack{s\le c\le t\\ t-s\searrow 0}} \omega_{s,t}(0) = 0$ and $\lim_{\substack{s\le c\le t\\ t-s\searrow 0}} \omega'_{s,t}(0) = 1$ for all $c \in [a,b]$.*

4. An Example of Discontinuous Evolution Family

Let $f : \mathbb{R} \to \mathbb{R}$ be an *additive function*, i.e., f satisfies

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in [0,\infty). \tag{1}$$

It is known that there exists a discontinuous additive function, sometimes called the *Hamel function*. For details of additive functions and the Hamel functions, see [Kuc09, pp.128–130]. Remark that it follows from (1) that $f(0) = 0$.

By using the Hamel function f, we define $\omega_{s,t}$ as

$$\omega_{s,t} := e^{if(t-s)} z \quad (z \in \mathbb{D},\ (s,t) \in [0,\infty)^2_+). \tag{2}$$

Then it is easy to see that the family $\{\omega_{s,t}\}_{[0,\infty)^2_+}$ is an evolution family, but not continuous with respect to both left and right parameters.

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