$\mathrm{PSL}(2,\mathbb{C})$-REPRESENTATIONS VIA TRIANGULATIONS IN DIMENSION 2 AND 3

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1. Introduction

$\mathrm{PSL}(2,\mathbb{C})$-representations of fundamental groups play an important role in low dimensional topology and geometry. In the 2-dimensional case, representations of surface groups into $\mathrm{PSL}(2,\mathbb{C})$ appear in the study of Kleinian groups, complex projective structures, Teichmüller spaces, and mapping class groups. In the 3-dimensional case, they are significant since many 3-manifolds admit hyperbolic structures, which give rise to discrete faithful representations in $\mathrm{PSL}(2,\mathbb{C})$. In this note, we give a parametrization of $\mathrm{PSL}(2,\mathbb{C})$-representations of a 3-manifold or surface group using ideal triangulations.

Thurston used ideal triangulations of 3-manifolds to show the existence of hyperbolic structures and analyze the deformation space of (incomplete) hyperbolic structures, especially for the figure eight knot complement. His method was systematically used by Neumann and Zagier to analyze the hyperbolic Dehn surgeries. In the 2-dimensional case, Penner gave a coordinate of the decorated Teichmüller space using ideal triangulations of a punctured surface [Pe]. His parametrization also works for $\mathrm{SL}(2,\mathbb{C})$-representations using complexified λ-length [NN].

In this note, we shall show how an ideal triangulation gives a parametrization of representations of a surface or 3-manifold group into $\mathrm{PSL}(2,\mathbb{C})$. Although the main statement Theorem 4.3 for the 3-dimensional case is well-known for experts, we explain here since it is useful to understand the 2-dimensional case and also it seems to be few reference in this generality. For the 2-dimensional case, our approach is different from Penner's work and it works even for closed surfaces. Our parametrization is an analogue of the complex Fenchel-Nielsen coordinates using ideal triangulations, and quite elementary and easy to give matrix representatives.

The exposition of this note has become more complicated than I intended. I recommend the reader to consult examples in Section 6 and 7 for the 2-dimensional case and Example 4.2 for the 3-dimensional case, in which I gave an explicit parametrization by matrix representatives.

2. Notations

We let $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I\}$ and $\mathrm{PGL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\mathbb{C}^*$. Since the square root is well-defined up to sign, we have a homomorphism

$$\mathrm{GL}(2,\mathbb{C}) \ni A \mapsto \frac{1}{\sqrt{|A|}} A \in \mathrm{PSL}(2,\mathbb{C}),$$

which induces an isomorphism $\mathrm{PGL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$. We sometimes use $\mathrm{PGL}(2,\mathbb{C})$ instead of $\mathrm{PSL}(2,\mathbb{C})$, because it usually simplifies the notation.

Let $\mathbb{H}^3$ be the hyperbolic 3-space. In this note, we only use the upper half space model $\mathbb{H}^3 = \{(x,y,t) \in \mathbb{R}^3 | t > 0\}$. The plane $\{(x,y,z) | t = 0\}$ can be compactified to the Riemann sphere $\mathbb{C}P^1$ and it can be regarded as an ideal boundary of $\mathbb{H}^3$. $\mathrm{PSL}(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ by linear fractional transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \mapsto \frac{az+b}{cz+d}.$$

This action extends to an isometry of $\mathbb{H}^3$. In fact, the group of orientation preserving isometries $\mathrm{Isom}^+(\mathbb{H}^3)$ is isomorphic to $\mathrm{PSL}(2,\mathbb{C})$. We simply call an element $g \in \mathrm{PSL}(2,\mathbb{C})$ *hyperbolic* if g has two fixed points on $\mathbb{C}P^1$, (so including loxodromic and elliptic in the usual definition). The following fact plays an important role in our description of $\mathrm{PSL}(2,\mathbb{C})$-representations.

Lemma 2.1. *There exists a unique element of* $\mathrm{PSL}(2,\mathbb{C})$ *which sends any distinct three points* (x_1,x_2,x_3) *of* $\mathbb{C}P^1$ *to the other distinct three points* (x'_1,x'_2,x'_3). *The matrix is given by*

$$\frac{\pm 1}{\sqrt{(x_1-x_2)(x_2-x_3)(x_3-x_1)(x'_1-x'_2)(x'_2-x'_3)(x'_3-x'_1)}} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where

$$\begin{aligned}
a_{11} &= x_1x'_1(x'_2-x'_3) + x_2x'_2(x'_3-x'_1) + x_3x'_3(x'_1-x'_2), \\
a_{12} &= x_1x_2x'_3(x'_1-x'_2) + x_2x_3x'_1(x'_2-x'_3) + x_3x_1x'_2(x'_3-x'_1), \\
a_{21} &= x_1(x'_2-x'_3) + x_2(x'_3-x'_1) + x_3(x'_1-x'_2), \\
a_{22} &= x_1x'_1(x_2-x_3) + x_2x'_2(x_3-x_1) + x_3x'_3(x_1-x_2).
\end{aligned}$$

Let M be a manifold. The set of all representations of $\pi_1(M)$ into $\mathrm{PSL}(2,\mathbb{C})$ is denoted by $R(M)$. A representation ρ is called *reducible* if $\rho(\pi_1(M))$ fixes a point of $\mathbb{C}P^1$. Otherwise it is called *irreducible.* The group $\mathrm{PSL}(2,\mathbb{C})$ acts on $R(M)$ by conjugation. Since the action is algebraic, we can define the algebraic quotient $X(M)$ of $R(M)$. This is called the *character variety* because it can be regarded as the set of the squares of the characters [HP]. If we restrict to the irreducible representations, $X(M)$ is nothing but the usual quotient by the action of $\mathrm{PSL}(2,\mathbb{C})$ ([Po], [CS]). See [HP], [BZ] and [MS] for details on $\mathrm{PSL}(2,\mathbb{C})$-character varieties.

3. ideal tetrahedra

An *ideal tetrahedron* is the convex hull of distinct 4 points of $\mathbb{C}P^1$ in $\mathbb{H}^3$. We assume that every ideal tetrahedron has an ordering on the vertices. Let z_0, z_1, z_2, z_3 be the

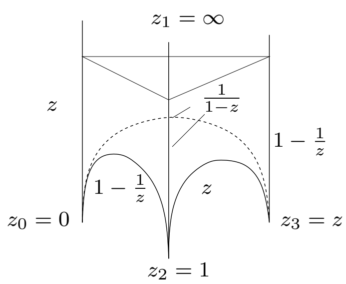

FIGURE 1. The complex parameters of the edges of an ideal tetrahedron.

vertices of an ideal tetrahedron. This ideal tetrahedron is parametrized by the cross ratio

$$[z_0 : z_1 : z_2 : z_3] = \frac{z_3 - z_0}{z_3 - z_1}\frac{z_2 - z_1}{z_2 - z_0} \in (\mathbb{C} - \{0, 1\}).$$

The cross ratio is invariant under the action of $\mathrm{PSL}(2,\mathbb{C})$, that is, $[gz_0 : gz_1 : gz_2 : gz_3] = [z_0 : z_1 : z_2 : z_3]$ for any $g \in \mathrm{PSL}(2,\mathbb{C})$. We denote the edge of the ideal tetrahedron spanned by z_i and z_j by $[z_i z_j]$. Take (i, j, k, l) to be an even permutation of $(0, 1, 2, 3)$. We define the complex parameter of the edge by the cross ratio $[z_i : z_j : z_k : z_l]$. This parameter only depend on the choice of the edge $[z_i z_j]$. We can easily observe that the opposite edge has same complex parameter and the other edges are parametrized by $\frac{1}{1-z}$ and $1 - \frac{1}{z}$ (see Figure 1).

Let g be a hyperbolic element whose fixed points are (x, y) and eigenvalues are e and e^{-1}. Then g is given by

$$g = \pm \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix} \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}^{-1} = \frac{\pm 1}{y - x} \begin{pmatrix} ey - e^{-1}x & -(e - e^{-1})xy \\ e - e^{-1} & -ex + e^{-1}y \end{pmatrix}. \tag{3.1}$$

To fix a parametrization of the eigenvalue e, we assume that x is the repelling fixed point and y is the attractive fixed point when $|e| > 1$. For example, $g = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}$ for $(x, y) = (0, \infty)$ and $g = \begin{pmatrix} e^{-1} & 0 \\ 0 & e \end{pmatrix}$ for $(x, y) = (\infty, 0)$. Let z be a point of $\mathbb{C}P^1$ distinct from x and y. Then the cross ratio $[x : y : z : gz]$ is equal to e^2 (Figure 2). Conversely for an ideal tetrahedron spanned by z_0, z_1, z_2, z_3, the element of $\mathrm{PSL}(2,\mathbb{C})$ which sends (z_0, z_1, z_2) to (z_0, z_1, z_3) has eigenvalues $(\sqrt{[z_0 : z_1 : z_2 : z_3]})^{\pm 1}$. So the cross ratio can be interpreted as the square of an eigenvalue of some matrix related to the ideal tetrahedron.

4. Ideal triangulation and representation of 3-manifold groups

In this note, a *triangulation* T is a cell complex obtained by gluing tetrahedra along their faces in pair by simplicial maps. We remark that this is not a simplicial complex since some vertices of a tetrahedron may be identified in T. In this note, we often distinguish between a 0-*simplex* of T and a *vertex* of a tetrahedron since various vertices of tetrahedra identified with a 0-simplex of T. We also distinguish between a 1-simplex of T and an edge of a tetrahedron. We denote the k-skeleton of T by $T^{(k)}$.

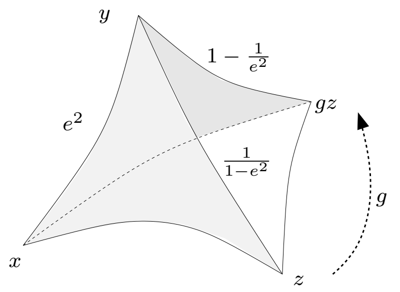

FIGURE 2. If g is a hyperbolic element whose fixed points are x and y, and eigenvalues are e and e^{-1}, then the edge (x, y) of the ideal tetrahedron (x, y, z, gz) has the complex parameter e^2.

Definition 4.1. A (topological) *ideal triangulation* of a compact 3-manifold M is a triangulation T such that $T - N(T^{(0)})$ is homeomorphic to M.

Here $N(T^{(0)})$ is a small open neighborhood of $T^{(0)}$. Because all 0-simplices are missing in M, we call them "ideal" points. In the usual definition of ideal triangulations, it is assumed that $\partial N(T^{(0)})$ consists of tori, but here we do not assume this property.

We denote the universal cover of M by $\widetilde{M}$. From now on, we construct an equivariant map $\widetilde{M} \to \mathbb{H}^3$ so called *developing map*, which gives rise to a $\mathrm{PSL}(2, \mathbb{C})$-representation. We assign an ordering on the vertices of each tetrahedron of T, then assign a complex parameter z_i for each tetrahedron. Pick a tetrahedron Δ of T, then put an ideal tetrahedron in $\mathbb{H}^3$ according to the complex parameter of Δ. Then the tetrahedra adjacent to Δ can be realized in $\mathbb{H}^3$ according to their complex parameters. Continuing in this way, we obtain a map from the universal cover of $T - T^{(1)}$ to $\mathbb{H}^3$. To obtain a map from the universal cover $\widetilde{M}$, which is homeomorphic to the universal cover of $T - N(T^{(0)})$, we have to impose the *gluing equation* around each 1-simplex of T. Consider the edges of tetrahedra which belong to a 1-simplex in T (Figure 3). When a path goes around the 1-simplex, we have to make sure that the developed image in $\mathbb{H}^3$ returns back to the same position. Since each edge has complex parameter z_i, $\frac{1}{1-z_i}$ or $1 - \frac{1}{z_i}$, we have to impose the following equation

$$\prod_{i=1} z_i^{p_{ji}} \left(\frac{1}{1-z_i}\right)^{p'_{ji}} \left(1 - \frac{1}{z_i}\right)^{p''_{ji}} = 1$$

for each 1-simplex (indexed by j) of T. These equations are simplified to the following form:

$$\pm \prod_{i=1} z_i^{r'_{ji}} (1 - z_i)^{r''_{ji}} = 1.$$

We call these equations *gluing equations*. Let

$$\mathcal{D}(M, T) = \{(z_1, \ldots, z_n) \in (\mathbb{C} - \{0, 1\})^n | \pm \prod_{i=1} z_i^{r'_{ji}} (1 - z_i)^{r''_{ji}} = 1 \quad (\forall j)\}.$$

When the triangulation is clear from the context we simply denote $\mathcal{D}(M, T)$ by $\mathcal{D}(M)$. For any point of $\mathcal{D}(M)$, we obtain a map $D : \widetilde{M} \to \mathbb{H}^3$. For any $\gamma \in \pi_1(M)$, there

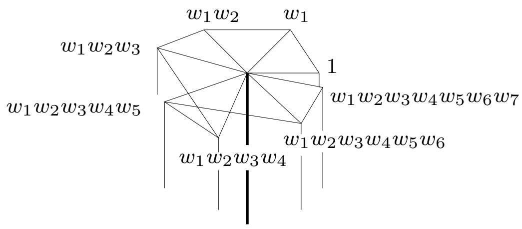

FIGURE 3. Developing map around a 1-simplex. Each w_k is one of z_i, $\frac{1}{1-z_i}$ or $1-\frac{1}{z_i}$.

exists a unique element $\rho(\gamma) \in \mathrm{PSL}(2,\mathbb{C})$ such that $D(\gamma p) = \rho(\gamma)D(p)$ for any $p \in \widetilde{M}$, where γ acts on $\widetilde{M}$ as a deck transformation. Then ρ is a homomorphism from $\pi_1(M)$ to $\mathrm{PSL}(2,\mathbb{C})$, which is called the *holonomy representation* of D. If we change the position of the first ideal tetrahedron Δ, we obtain a conjugate representation. So we have a map $\mathcal{D}(M) \to X(M)$ by sending an element of $\mathcal{D}(M)$ to its holonomy representations.

Example 4.2. The complement of the figure eight knot K can be decomposed into two ideal tetrahedra (Figure 4 and 5). As indicated in Figure 5, we assign complex parameters x and y for these two tetrahedra. There exist two 1-simplices in this ideal triangulation. The gluing equations of these 1-simplices coincide and given by

$$xy(1-x)(1-y) = 1. \tag{4.1}$$

Let x_1, x_2 and x_3 be the elements of $\pi_1(S^3-K)$ as indicated in Figure 4, then the relations are given by

$$\pi_1(S^3-K) \cong \langle x_1, x_2, x_3 | x_3x_2x_3^{-1}x_1^{-1} = 1, \quad x_2^{-1}x_1x_2x_1^{-1}x_3^{-1} = 1\rangle.$$

Realize an ideal tetrahedron parametrized by x in $\mathbb{H}^3$ as the convex hull of $(0,\infty,1,x)$. Then adjacent ideal tetrahedra parametrized by y is developed to $(0,\infty,x,xy)$ (Figure 5). We denote this pair of two ideal tetrahedra by P, which plays the role of a fundamental domain. The holonomy representation is obtained as follows. For each x_i, there exists a pair of faces corresponding to x_i. Let $\rho(x_i)$ be the matrix which sends one face of the pair to the other face (Figure 5). By Lemma 2.1, there exists a unique such element of $\mathrm{PSL}(2,\mathbb{C})$. For example $\rho(x_1)$ is the matrix which sends $(0,\infty,xy)$ to $(0,1,x)$ and $\rho(x_2)$ sends (∞,x,xy) to $(\infty,1,0)$. Explicitly these are given by

$$\rho(x_1) = \frac{\pm 1}{\sqrt{y(1-x)}}\begin{pmatrix} 1 & 0 \\ 1 & y(1-x) \end{pmatrix},$$
$$\rho(x_2) = \frac{\pm 1}{\sqrt{x(1-y)}}\begin{pmatrix} 1 & -xy \\ 0 & x(1-y) \end{pmatrix}.$$

If (x,y) satisfies the gluing equation (4.1), this is a homomorphism from $\pi_1(S^3-K)$ to $\mathrm{PSL}(2,\mathbb{C})$.

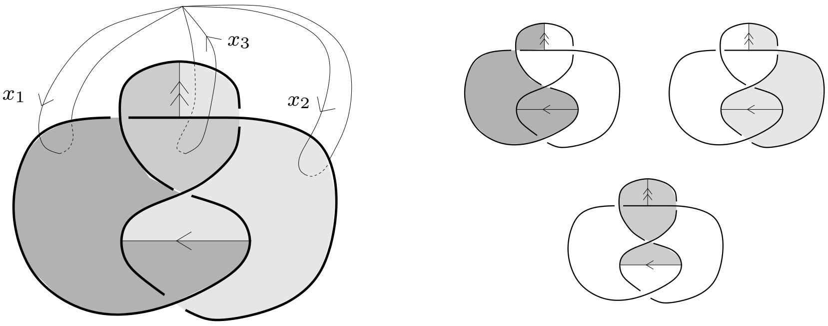

FIGURE 4. The figure eight knot and 2-simplices of an ideal triangulation of the complement.

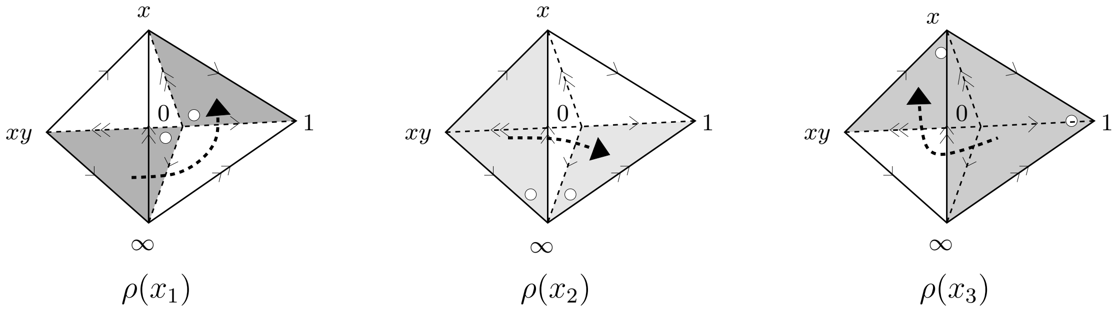

FIGURE 5. Maps corresponding to the generators.

Next we discuss the restriction of the holonomy representation to a boundary subgroup. Each boundary component of M is expressed as $\partial N(p)$ by some 0-simplex p of T. Therefore the restriction of the holonomy representation to the boundary subgroup $\partial N(p)$ fixes a point of $\mathbb{C}P^1$, i.e. reducible. The result we have obtained so far summarized as follows:

Theorem 4.3. *Let M be a compact 3-manifold and T be an ideal triangulation of M. For $(z_1, \dots, z_n) \in \mathcal{D}(M)$, there exists a $\mathrm{PSL}(2,\mathbb{C})$ representation of $\pi_1(M)$ up to conjugation. This gives an algebraic map $\mathcal{D}(M) \to X(M)$. The restriction of this representation to any boundary subgroup is reducible.*

If the boundary components consist of tori, such representations are generic since any boundary subgroup is abelian.

Since monodromies restrict to a boundary subgroup behave in a commutative fashion, we can easily compute them. Let p be a 0-simplex of T. Now $\partial N(p)$ is triangulated by the truncated vertices (Figure 6). By conjugation, we assume that p is developed to ∞. Then the truncated vertices are developed into Euclidean plane as triangles. By the observation in the previous section, each triangle can be interpreted as a matrix fixing ∞ and whose eigenvalues are given by the complex parameter. For example the monodromy of the path indicated in Figure 6 is given by

$$\begin{pmatrix} \sqrt{w_1} & * \\ 0 & 1/\sqrt{w_1} \end{pmatrix} \begin{pmatrix} 1/\sqrt{w_2} & * \\ 0 & \sqrt{w_2} \end{pmatrix} \begin{pmatrix} \sqrt{w_3} & * \\ 0 & 1/\sqrt{w_3} \end{pmatrix} \begin{pmatrix} 1/\sqrt{w_4} & * \\ 0 & \sqrt{w_4} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{w_1 w_3}{w_2 w_4}} & * \\ 0 & \sqrt{\frac{w_2 w_4}{w_1 w_3}} \end{pmatrix}$$

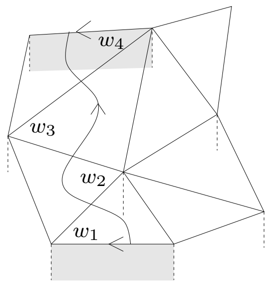

FIGURE 6. Developing map around a 0-simplex.

Therefore this matrix has eigenvalues $\sqrt{\frac{w_1 w_3}{w_2 w_4}}^{\pm 1}$. In this way, the squares of the eigenvalues of the monodromy along a boundary curve has the form $\pm \prod_i z_i^{m_i'}(1 - z_i)^{m_i''}$ by some integers m_i' and m_i''. This fact is useful to find complete hyperbolic structures of knot complements, and also useful for hyperbolic Dehn surgeries ([Th], [We]).

5. Representations of surface groups

In this section we give a parametrization of the representations of a surface group based on ideal triangulations. Our construction divided into two steps. First, we construct a parametrization of the representations of the fundamental group of a pair of pants. Next we describe how these representations are glued along a simple closed curve. As in the previous section, we will use developing maps to describe these representations.

5.1. **Pants decomposition.** Let $S = S_{g,n}$ be a surface of genus g with n holes. In the following, we assume that the Euler characteristic of S is negative $(2 - 2g - n < 0)$. A *pants decomposition* C is a disjoint union of simple closed curve on S such that the complementary region is a collection of three holed spheres (pairs of pants). The number of simple closed curves of C is equal to $3g - 3 + n$. In this note, we assume that each simple closed curve of a pants decomposition is oriented. To emphasize the orientation, we denote an oriented simple closed curve as $\overrightarrow{c}$. We fix a pants decomposition $C = \overrightarrow{c}_1 \cup \cdots \cup \overrightarrow{c}_{3g-3+n}$. Let ρ be a representation $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ satisfying the following two conditions:

- $\rho(\overrightarrow{c}_i)$ is a hyperbolic element for any $i = 1, \dots, 3g - 3 + n$,
- the restriction $\rho|_{\pi_1(P)}$ is an irreducible representation for any pair of pants $P \subset S - C$.

We will parametrize the representations satisfying these two conditions. We remark that the set of all such representations is a codimension zero subset of $X(S)$. Moreover there exists a pants decomposition satisfying these two conditions for any non-elementary representation [GKM].

5.2. **Representations of a pair of pants.** Fix a hyperbolic metric on S, so we can regard the universal covering of S as the Poincaré disk model $\mathbb{H}^2$. Let $p : \mathbb{H}^2 \to S$ be the projection map. Each simple closed curve of C is represented by a geodesic curve with respect to this hyperbolic metric. Let P be a pair of pants of $S - C$. Permuting indices,

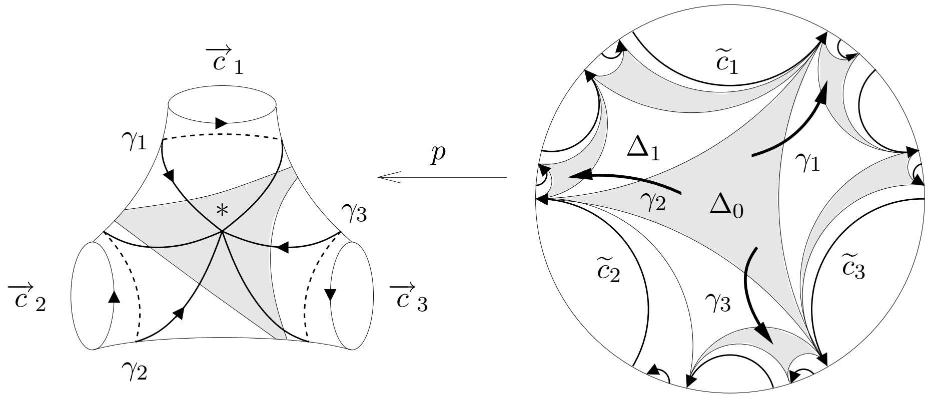

FIGURE 7. The lift of a pair of pants to the Poincaré disk. Here $\widetilde{c_i}$ is a lift of c_i. Three arrows emanating from Δ_0 mean deck transformations corresponding to $\gamma_1, \gamma_2, \gamma_3 \in \pi_1(P, *)$ respectively.

we assume that $\overrightarrow{c}_1$, $\overrightarrow{c}_2$ and $\overrightarrow{c}_3$ are on the boundary of P (two of them may coincide). We triangulate P by two ideal triangles whose ideal vertices go to the directions of $\overrightarrow{c}_i$. Let $\widetilde{P}$ be the inverse image of P in $\mathbb{H}^2$. $\widetilde{P}$ has infinitely many geodesic boundaries and they are lifts of some $\overrightarrow{c}_i$. The ideal triangulation of P lifts to a triangulation of $\widetilde{P}$. We fix one ideal triangle Δ_0 of $\widetilde{P}$. Let Δ_1 be an ideal triangle adjacent to Δ_0 in $\widetilde{P}$. Δ_0 and Δ_1 form a fundamental domain of $\pi_1(P, *)$, where $*$ is a base point on $p(\Delta_0)$. By deck transformations, there exists a one to one correspondence between $\pi_1(P, *)$ and the lifts of $p(\Delta_0)$. Let γ_i be a loop based at $*$ and going around $\overrightarrow{c}_i$. Then $\pi_1(P, *)$ is a free group generated by any two of γ_1, γ_2, γ_3. As a deck transformation, γ_i sends Δ_0 to a nearest lift of $p(\Delta_0)$ (Figure 7).

We shall construct a developing map $D : \widetilde{P} \to \mathbb{H}^3$ of ρ. Since $\rho(\gamma_i)$ is a hyperbolic element by assumption, $\rho(\gamma_i)$ has two fixed points on $\mathbb{C}P^1$. Let x_i be one of the two fixed point of $\rho(\gamma_i)$. Since $\rho|_{\pi_1(P)}$ is irreducible, x_1, x_2 and x_3 are mutually distinct. Send $\Delta_0 \subset \widetilde{P}$ to the ideal triangle (x_1, x_2, x_3). Then develop the ideal triangle in $\mathbb{H}^3$ by the action of $\rho(\pi_1(P))$. The ideal triangles $\rho(\pi_1(P))(x_1, x_2, x_3)$ automatically determine the image of $\pi_1(P)\Delta_1$. So we have obtained a ρ-equivariant map $D : \widetilde{P} \to \mathbb{H}^3$. Since the transformation that sends $D(\Delta_0)$ to $D(\gamma\Delta_0)$ is uniquely determined by Lemma 2.1, we can conversely construct ρ from the developing map D.

From now on we observe that the developing map D is uniquely determined by the eigenvalues of $\rho(\gamma_i)$ up to conjugation. First of all, we assume that $x_1 = 0$,$x_2 = \infty$ and $x_3 = 1$. Let y_i be the other fixed point of $\rho(\gamma_i)$ and $e_i^{\pm 1} \in \mathbb{C} - \{0, \pm 1\}$ be the eigenvalues. Then ρ is given by

$$\rho(\gamma_1) = \begin{pmatrix} e_1^{-1} & 0 \\ \frac{e_1^{-1} - e_1}{y_1} & e_1 \end{pmatrix}, \quad \rho(\gamma_2) = \begin{pmatrix} e_2 & (e_2^{-1} - e_2)y_2 \\ 0 & e_2^{-1} \end{pmatrix},$$
$$\rho(\gamma_3) = \frac{1}{y_3 - 1} \begin{pmatrix} e_3^{-1}y_3 - e_3 & (e_3 - e_3^{-1})y_3 \\ e_3^{-1} - e_3 & e_3 y_3 - e_3^{-1} \end{pmatrix}$$

From the identity $\rho(\gamma_1)\rho(\gamma_2)\rho(\gamma_3) = I$, we have

$$y_1 = \frac{e_1 - e_1^{-1}}{e_1 - e_2 e_3^{-1}}, \quad y_2 = \frac{e_2^{-1} - e_1^{-1} e_3}{e_2^{-1} - e_2}, \quad y_3 = \frac{e_1 - e_2 e_3}{e_1 - e_2 e_3^{-1}}.$$

Therefore ρ is uniquely determined by the eigenvalues $e_i^{\pm 1}$ of $\rho(\gamma_i)$. This gives a lift of the $\mathrm{PSL}(2,\mathbb{C})$-representation to a $\mathrm{SL}(2,\mathbb{C})$-representation. Any other lift is obtained by the action of $H^1(P;\mathbb{Z}/2\mathbb{Z})$, $(e_1, e_2, e_3) \mapsto (\epsilon_1 e_1, \epsilon_2 e_2, \epsilon_3 e_3)$, where $\epsilon_i = \pm 1$ satisfying $\epsilon_1\epsilon_2\epsilon_3 = 1$.

The general case is as follows:

Proposition 5.1. *Let $\rho : \pi_1(P) \to \mathrm{PSL}(2,\mathbb{C})$ be an irreducible representation such that $\rho(\gamma_i)$ is hyperbolic. Let x_i be one of the fixed points of $\rho(\gamma_i)$ and $e_i^{\pm}$ be the eigenvalues. Then ρ is given by*

$$\rho(\gamma_i) = \frac{1}{e_i e_{i+2}(x_{i+1} - x_i)(x_{i+2} - x_i)} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

Interchange i+1 and i+2

$$\begin{aligned} a_{11} &= e_i^2 e_{i+2} x_i (x_i - x_{i+1}) + e_{i+2} x_{i+1} (x_{i+2} - x_i) + e_i e_{i+1} x_i (x_{i+1} - x_{i+2}), \\ a_{12} &= x_i (e_i^2 e_{i+2} x_{i+2} (x_{i+1} - x_i) + e_{i+2} x_{i+1} (x_i - x_{i+2}) + e_i e_{i+1} x_i (x_{i+2} - x_{i+1})), \\ a_{21} &= e_i^2 e_{i+2} (x_i - x_{i+1}) + e_{i+2} (x_{i+2} - x_i) + e_i e_{i+1} (x_{i+1} - x_{i+2}), \\ a_{22} &= e_i^2 e_{i+2} x_{i+2} (x_{i+1} - x_i) + e_{i+2} x_i (x_i - x_{i+2}) + e_i e_{i+1} x_i (x_{i+2} - x_{i+1}), \end{aligned} \tag{5.1}$$

up to the action of $H^1(P;\mathbb{Z}/2\mathbb{Z})$. The other fixed points of $\rho(\gamma_i)$ is given by

$$y_i = \frac{e_i^2 e_{i+2} x_{i+2} (x_i - x_{i+1}) + e_{i+2} x_{i+1} (x_{i+2} - x_i) + e_i e_{i+1} x_i (x_{i+1} - x_{i+2})}{e_i^2 e_{i+2} (x_i - x_{i+1}) + e_{i+2} (x_{i+2} - x_i) + e_i e_{i+1} (x_{i+1} - x_{i+2})}. \tag{5.2}$$

5.3. **Gluing developing maps of pairs of pants.** In this subsection, we discuss how to glue two developing maps of pairs of pants along a common geodesic. Let P and P' be pairs of pants adjacent along a simple closed curve $\overrightarrow{c}$ (P and P' may coincide). Let $\widetilde{c}$ be a lift of $\overrightarrow{c}$. We will glue the developing maps of P and P' along $\widetilde{c}$. Let ∞ be the head endpoint of $\widetilde{c}$. Take an adjacent pair of ideal triangles Δ_0 and Δ_1 in $\widetilde{P}$ sharing ∞ as a common ideal vertex. We also take Δ_0' and Δ_1' in $\widetilde{P'}$ similarly (Figure 8). Then Δ_0 and Δ_1 are developed by $\rho(\pi_1(P))$ and Δ_0' and Δ_1' by $\rho(\pi_1(P'))$ as in the previous subsection.

We denote by Γ the boundary subgroup of $\pi_1(P)$ (also $\pi_1(P')$) corresponding to $\overrightarrow{c}$. Now $\Delta_0 \cup \Delta_1$ forms an ideal quadrilateral in $\widetilde{P}$, two of its edges have ideal vertices ∞. The developed image of these two edges are in the same orbit under the action of $\rho(\Gamma)$. Let Δ be the ideal triangle whose vertices are ∞ and the endpoints of these two edges other than ∞ (see Figure 8). We also define the ideal triangle Δ' in $\widetilde{P'}$ in a similar way. Now $\Gamma\Delta$ and $\Gamma\Delta'$ are developed in $\rho(\Gamma)$-equivariant way. We define the *twist parameter* with respect to $\overrightarrow{c}$ by the eigenvalue of the hyperbolic element which sends Δ to Δ', such eigenvalue is uniquely determined up to sign by the convention of (3.1). We remark that the twist parameter depends on the choice of ideal triangles Δ_0, Δ_1 of $\widetilde{P}$ and Δ_0', Δ_1' of $\widetilde{P'}$, like twist parameters of Fenchel-Nielsen coordinates. Via the equivariant map between $\Gamma\Delta$ and $\Gamma\Delta'$, we can glue two developing maps along $\widetilde{c}$. Conversely if two pairs of pants have the boundary curves with same holonomy, we can glue the developing maps twisted by any non-zero complex number.

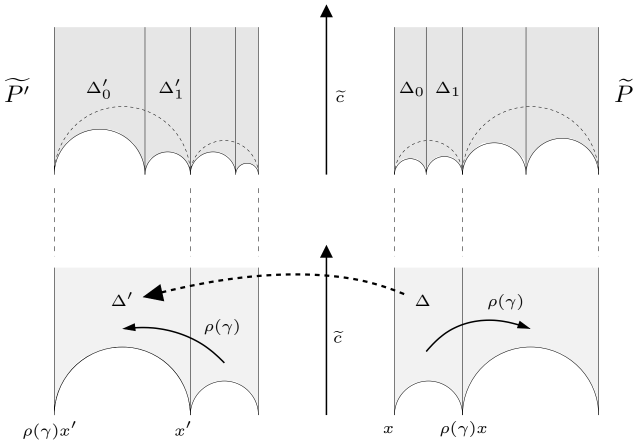

FIGURE 8. The gluing of the developing map. Here γ is a generator of the boundary subgroup corresponding to c.

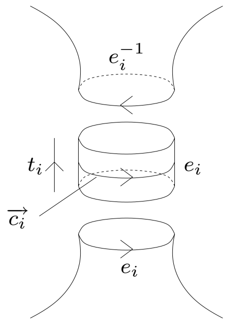

FIGURE 9

Continuing this process to all pairs of pants, we obtain a parametrization of a subset $X(S)$ satisfying the two conditions of subsection 5.1 by $6g-6+n$ complex numbers, $3g-3+n$ of which are eigenvalues and the rest are twist parameters.

5.4. **Coordinates of surface representations.** In this subsection, we give a coordinate of surface representations satisfying the two conditions of subsection 5.1. Let $S = S_{g,n}$ be a surface and $C = \overrightarrow{c}_1 \cup \cdots \cup \overrightarrow{c}_{3g-3+n}$ be a pants decomposition by oriented curves. For any curve $\overrightarrow{c}_i$, we assign two complex parameters, the eigenvalue e_i and the twist parameter t_i. To fix a parametrization, we orient the boundaries of each pair of pants counter-clockwise viewing from above as shown in Figure 7. Now each boundary curve of a pair of pants inherits the eigenvalue parameter e_i or e_i^{-1} (Figure 9). We fix a twist parameter in the right handed screw direction with respect to $\overrightarrow{c}_i$ (Figure 9).

Fix one pair of pants P of $S - C$. Let $\overrightarrow{d}_1$, $\overrightarrow{d}_2$ and $\overrightarrow{d}_3$ be the boundary curves of P with the same orientations as Figure 7. If we fix a set of fixed points corresponding to $\overrightarrow{d}_i$, the developing map of $\widetilde{P}$ is uniquely determined by (5.1). We will show how these fixed points transit to an adjacent pair of pants by the twist parameter.

Let e_i be the eigenvalue corresponding to $\overrightarrow{d}_i$ and x_i be the attractive fixed point when $|e_i| > 1$. Then (e_1^{-1}, e'_2, e'_3) be the eigenvalues of the pair of pants adjacent along

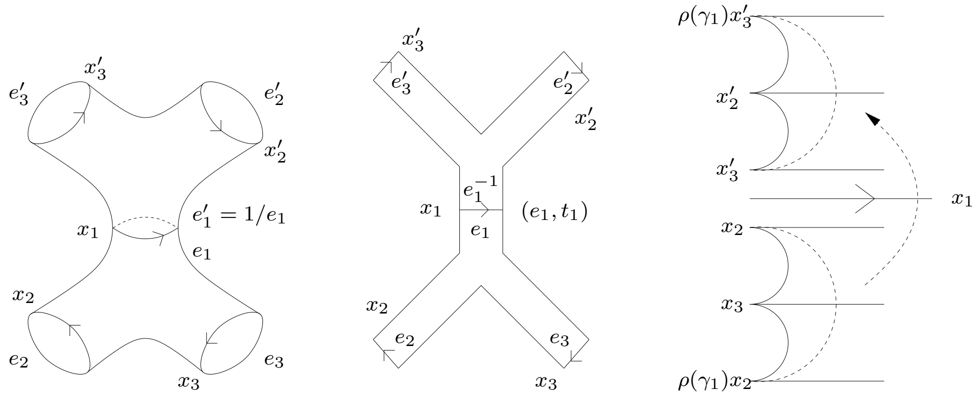

FIGURE 10. Transition from (x_1, x_2, x_3) to (x_1, x_2', x_3'). Here the matrix which sends $(x_1, x_2, \rho(\gamma_1)x_2)$ to $(x_1, x_3', \rho(\gamma_1)x_3')$ has eigenvalues $t_1^{\pm 1}$.

$\overrightarrow{d}_1$ (Figure 10). We will compute the fixed points x_2' and x_3' corresponding to e_2' and e_3' respectively. To encode these information, it is convenient to represent the pants decomposition by a trivalent fat graph with directed edges (the middle of the Figure 10). Here the ideal triangle $(x_1, x_2, \rho(\gamma_1)x_2)$ is maps to $(x_1, x_3', \rho(\gamma_1)x_3')$ by

$$\pm \begin{pmatrix} x_1 & y_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ 1 & 1 \end{pmatrix}^{-1},$$

(see (3.1)) where y_1 is the repelling fixed point corresponding to e_i, which can be computed by (5.2). In this way we can compute x_3'. Then we can also compute x_2' by considering the cross ratio of the ideal tetrahedron $(x_1, x_3', x_2', \rho(\gamma_1)x_3')$. Finally we have

$$\begin{aligned} x_2' &= \frac{a_1}{a_2}, \\ a_1 &= e_1((e_2 - e_1e_3)(e_3' - e_1e_2'){t_1}^2 + e_3(e_1e_3' - e_2'))x_1(x_2 - x_3) \\ &\quad + {e_1}^2e_2(e_1e_3' - e_2')x_2(x_3 - x_1) + e_2(e_1e_3' - e_2')x_3(x_1 - x_2), \\ a_2 &= e_1((e_2 - e_1e_3)(e_3' - e_1e_2'){t_1}^2 + e_3(e_1e_3' - e_2'))(x_2 - x_3) \\ &\quad + {e_1}^2e_2(e_1e_3' - e_2')(x_3 - x_1) + e_2(e_1e_3' - e_2')(x_1 - x_2), \end{aligned} \tag{5.3}$$

$$x_3' = \frac{((e_2 - e_1e_3){t_1}^2 + e_1e_3)x_1(x_2 - x_3) + {e_1}^2e_2x_2(x_3 - x_1) + e_2x_3(x_1 - x_2)}{((e_2 - e_1e_3){t_1}^2 + e_1e_3)(x_2 - x_3) + {e_1}^2e_2(x_3 - x_1) + e_2(x_1 - x_2)} \tag{5.4}$$

Remark 5.2. We remark that Bonahon gave a parametrization of $\mathrm{PSL}(2,\mathbb{C})$ representations of the fundamental group of a surface by using the *shear-bend cocycle* of maximal geodesic lamination λ in section 10 of [Bo]. Our parametrization closely related to the shear-bend cocycle. In our case the ideal triangulation associated to pants decomposition gives a maximal geodesic lamination of a surface.

6. Once punctured torus

In this section, we give explicit representations parametrized by (e_i, t_i) for $S_{1,1}$. The surface $S_{1,1}$ decomposed into one pair of pants. We give the eigenvalue parameters e_1, e_2,

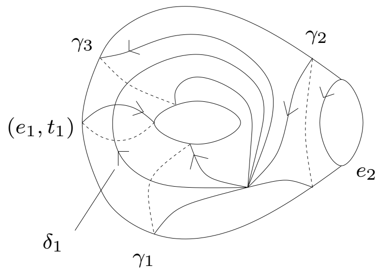

FIGURE 11. One holed torus.

and the twist parameter t_1 as in Figure 11. Define the based loops γ_1, γ_2, γ_3 and δ_1 as indicated in Figure 11. Then we let the fixed points $(x_1, x_2, x_3) = (\infty, 0, 1)$. By (5.1), we have

$$\begin{aligned}
\rho(\gamma_1) &= \begin{pmatrix} e_1 & e_1^{-1} - e_1^{-1}e_2^{-1} \\ 0 & e_1^{-1} \end{pmatrix}, \\
\rho(\gamma_2) &= \begin{pmatrix} e_2^{-1} & 0 \\ e_1^2 - e_2 & e_2 \end{pmatrix}, \\
\rho(\gamma_3) &= \begin{pmatrix} e_1^{-1}e_2 & e_1^{-1} - e_1^{-1}e_2 \\ e_1^{-1}e_2 - e_1 & e_1 + e_1^{-1} - e_1^{-1}e_2 \end{pmatrix}.
\end{aligned}$$

This satisfies $\rho(\gamma_1)\rho(\gamma_2)\rho(\gamma_3) = I$. We apply (5.3) and (5.4) for $(x_1, x_2, x_3) = (\infty, 0, 1)$ and $(e_1, e_2, e_3) = (e_1, e_2, e_1^{-1})$ and $(e_1', e_2', e_3') = (e_1^{-1}, e_1, e_2)$. Then we have

$$\begin{aligned}
x_2' &= \left(\frac{e_2 - e_1^2}{e_2(e_1^2 - 1)}\right) t_1^2 + \frac{1 - e_2}{e_2(e_1^2 - 1)}, \\
x_3' &= \frac{(t_1^2 - 1)(e_2 - 1)}{e_2(e_1^2 - 1)}
\end{aligned}$$

Because $\rho(\delta_1)$ is the matrix which sends $(\infty, 0, 1)$ to (x_3', x_2', ∞), we have

$$\rho(\delta_1) = \begin{pmatrix} (e_1^2 - e_2)t_1^2 + (e_2 - 1) & (t_1^2 - 1)(e_2 - 1) \\ -e_2(e_1^2 - 1) & e_2(e_1^2 - 1) \end{pmatrix}$$

in $\mathrm{PGL}(2, \mathbb{C})$. Actually these matrices satisfies the equality

$$\rho(\delta_1)^{-1}\rho(\gamma_1)^{-1}\rho(\delta_1)\rho(\gamma_1) = \rho(\gamma_2)^{-1}.$$

When $e_2 = -1$, after normalizing the matrices to $\mathrm{SL}(2, \mathbb{C})$, we have

$$\begin{aligned}
\rho(\gamma_1) &= \begin{pmatrix} e_1 & 2e_1^{-1} \\ 0 & e_1^{-1} \end{pmatrix}, \\
\rho(\delta_1) &= \frac{1}{\sqrt{-1}t_1(1 - e_1^2)} \begin{pmatrix} (e_1^2 + 1)t_1^2 - 2 & -2(t_1^2 - 1) \\ e_1^2 - 1 & -(e_1^2 - 1) \end{pmatrix}.
\end{aligned}$$

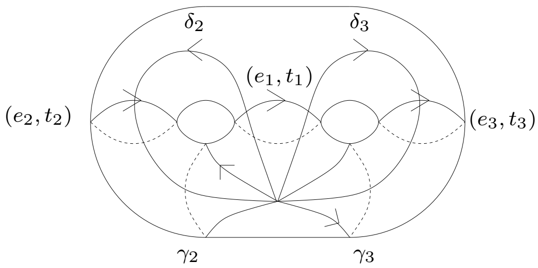

FIGURE 12. Genus 2 surface.

Replace t_1 with $\sqrt{-1}t_1$,

$$\rho(\gamma_1) = \begin{pmatrix} e_1 & 2e_1^{-1} \\ 0 & e_1^{-1} \end{pmatrix},$$

$$\rho(\delta_1) = \frac{1}{t_1(1-e_1^2)} \begin{pmatrix} -(e_1^2+1)t_1^2 - 2 & 2(t_1^2+1) \\ e_1^2 - 1 & -(e_1^2-1) \end{pmatrix} = \begin{pmatrix} \frac{-(e_1^2+1)t_1^2-2}{t_1(1-e_1^2)} & \frac{2(t_1^2+1)}{t_1(1-e_1^2)} \\ -t_1^{-1} & t_1^{-1} \end{pmatrix}.$$

Let $A = \rho(\gamma_1)$ and $B = \rho(\delta_1)$. The traces of A, B and AB are given by

$$\mathrm{tr}(A) = e_1 + e_1^{-1}, \quad \mathrm{tr}(B) = \frac{(e_1 + e_1^{-1})(t_1 + t_1^{-1})}{(e_1 - e_1^{-1})},$$
$$\mathrm{tr}(AB) = \frac{(e_1 + e_1^{-1})(e_1 t_1 + e_1^{-1} t_1^{-1})}{(e_1 - e_1^{-1})}.$$

Clearly this triple satisfy the Markov identity

$$\mathrm{tr}(A)^2 + \mathrm{tr}(B)^2 + \mathrm{tr}(AB)^2 - \mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(AB) = 0.$$

7. Genus 2 surface

We give explicit representations parametrized by (e_i, t_i) for a closed surface $S_{2,0}$. Let (e_i, t_i) be the parameters and γ_i and δ_i be the based loops as indicated in Figure 12. Let $\overrightarrow{c}_i$ be the oriented closed curve corresponding to the parameters (e_i, t_i). We fix the fixed points of the lower pants by $(x_1, x_2, x_3) = (\infty, 0, 1)$. The matrices $\rho(\gamma_2)$ and $\rho(\gamma_3)$ are immediately obtained by (5.1). Then we compute the ideal triangle (x_1, x_2', x_3') opposite to (x_1, x_2, x_3) with respect to $\overrightarrow{c}_1$ by using (5.3) and (5.4). We also compute the ideal triangle (x_1'', x_2', x_3'') opposite to (x_1, x_2', x_3') with respect to $\overrightarrow{c}_2$. Now the map which sends $(\infty, 0, 1)$ to (x_1'', x_2', x_3'') gives the matrix $\rho(\delta_2)$. Similarly we can compute the matrix $\rho(\delta_3)$.

After extensive calculations we obtain an explicit representation into $\mathrm{SL}(2,\mathbb{C})$ as follows:

$$\begin{aligned}
\rho(\gamma_2) &= \begin{pmatrix} e_2^{-1} & 0 \\ -e_2 + e_1 e_3^{-1} & e_2 \end{pmatrix}, \\
\rho(\gamma_3) &= \begin{pmatrix} e_1^{-1} e_2 & e_3 - e_1^{-1} e_2 \\ -e_3^{-1} + e_1^{-1} e_2 & e_3 + e_3^{-1} - e_1^{-1} e_2 \end{pmatrix}, \\
\rho(\delta_2) &= \frac{1}{t_1 t_2} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \\
&\quad a_{11} = \frac{(e_1e_2e_3 - 1)(e_1e_2 - e_3){t_1}^2{t_2}^2 + (e_2e_3 - e_1)(e_1e_3 - e_2)({t_1}^2 + {t_2}^2 - 1)}{({e_1}^2 - 1)({e_2}^2 - 1)e_3}, \\
&\quad a_{12} = -\frac{(e_1e_3 - e_2)({t_1}^2 - 1)}{({e_1}^2 - 1)e_2}, \\
&\quad a_{21} = \frac{e_2({t_2}^2 - 1)(e_2e_3 - e_1)}{({e_2}^2 - 1)e_3}, \\
&\quad a_{22} = 1, \\
\rho(\delta_3) &= \frac{1}{e_1({e_3}^2 - 1)t_1 t_3} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \\
&\quad b_{11} = -(e_1(e_1e_2e_3 - 1)(e_1e_3 - e_2){t_1}^2{t_3}^2 + e_1(e_2e_3 - e_1)(e_1e_2 - e_3){t_1}^2 \\
&\qquad\qquad + (e_2e_3 - e_1)(e_1e_3 - e_2)(1 - {t_3}^2))/(({e_1}^2 - 1)e_2), \\
&\quad b_{12} = (e_1e_3 - e_2)(e_1(e_1e_2e_3 - 1){t_1}^2{t_3}^2 + (e_1 - e_2e_3){t_3}^2 \\
&\qquad\qquad + e_1e_3(e_3 - e_1e_2){t_1}^2 + e_3(e_2 - e_1e_3))/(({e_1}^2 - 1)e_2), \\
&\quad b_{21} = (e_2e_3 - e_1)({t_3}^2 - 1), \\
&\quad b_{22} = (-e_2e_3 + e_1){t_3}^2 - e_1{e_3}^2 + e_2e_3.
\end{aligned}$$

Actually we can check that these matrices satisfy the equality

$$\rho(\delta_2)\rho(\gamma_2)^{-1}\rho(\delta_2)^{-1}\rho(\gamma_2)\rho(\gamma_3)\rho(\delta_3)\rho(\gamma_3)^{-1}\rho(\delta_3)^{-1} = I,$$

for any $(e_1, e_2, e_3, t_1, t_2, t_3) \in (\mathbb{C} - \{0, \pm 1\})^3 \times (\mathbb{C} - \{0\})^3$.

Remark 7.1. There exist two connected components of $X(S)$, one corresponds to representations liftable to $\mathrm{SL}(2,\mathbb{C})$ and the other does not [Go]. Since we have used (5.1), which gives liftable representations on each pair of pants, we could parametrize the liftable component. If we choose $\mathrm{SL}(2,\mathbb{C})$ representatives to satisfy $\rho(\gamma_1)\rho(\gamma_2)\rho(\gamma_3) = -I$ on some pairs of pants, then the representations might not be liftable. If the Stiefel-Whitney class w_2 is non-trivial, this gives a parametrization of the non-liftable component.

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