Stochastic Loewner Evolutions and Statistical Mechanics

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Abstract: This is a review talk on the stochastic Loewner evolution (SLE) from the view point of statistical physics. First we explain four important models of statistical mechanics defined on planar lattices and their possible continuum (scaling) limits. Then SLE_κ is introduced, where $\kappa > 0$ is a parameter indicating the variance of the one-dimensional Brownian motion randomly driving the Loewner chain, and basic properties of SLE_κ are listed up. Finally the correspondence between the continuum (scaling) limits of the statistical mechanics models and SLE_κ with special values of κ is discussed.

1 Statistical Mechanics Models and Measures on Continuous Path Space

1.1 Scaling limits of planar lattice models

On the complex plane $\mathbb{C}$, put the square lattice $\mathbf{S} = \mathbb{Z} \times \sqrt{-1}\mathbb{Z}$. A lattice path of length $n \in \mathbb{N}$, $\omega = (\omega(0), \dots, \omega(n))$ is defined as a sequence of $n+1$ vertices $\omega(i) \in \mathbf{S}, 0 \leq i \leq n$ such that $|\omega(i) - \omega(i-1)| = 1, 1 \leq i \leq n$. The (simple and symmetric) random walk model is a statistical ensemble of lattice paths uniformly distributed in a set of lattice paths with a given length. For $z \in \mathbf{S}, n \in \mathbb{N}$, let W_n^z be the set of all lattice paths of length n starting from z, *i.e.* $\omega(0) = z$. Since $|W_n^z| = 4^n$, the measure (weight) of each lattice path $\omega \in W_n^z$ is given by 4^{-n}. We consider a collection of ensembles with different lengths of lattice paths and call the element of the collection simply a random walk (RW).

Here we consider a square-shaped domain $D_0 = \{x + \sqrt{-1}y : -1 < x < 1, 0 < y < 2\}$, in $\mathbb{C}$ and specify the two points on its boundary, $O = 0$ (the origin) and $P = 2\sqrt{-1}$. (In this article, a domain means a connected, open subset of $\mathbb{C}$.) Then choose an integer $N \in \mathbb{N}$ and magnify the domain D_0 by factor N with the origin fixed. Let $\Omega_N(D_0; O, P)$ be the collection of all RWs $\{\omega\}$ starting from $NO = 0$ (*i.e.* $\omega(0) = 0$) and terminating at $NP = 2N\sqrt{-1}$ (*i.e.* $\omega(|\omega|) = 2N\sqrt{-1}$) such that $\omega \setminus \{\omega(0), \omega(|\omega|)\} \subset ND_0$, where the length of each lattice path ω is denoted by $|\omega|$. The total mass of this measure is given by

$$Z_N(D_0; O, P) = \sum_{\omega \in \Omega_N(D_0; O, P)} 4^{-|\omega|}. \tag{1.1}$$

Such quantity is called the partition function in the statistical mechanics. We can show that it decays as

$$Z_N(D_0; O, P) \sim C(D_0; O, P) N^{-2}, \quad N \to \infty, \tag{1.2}$$

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where $f(N) \sim g(N), N \to \infty$ means $f(N)/g(N) \to 1, N \to \infty$. The coefficient $C(D_0; O, P)$ is given by the normal derivative at the origin $O \in \partial D_0$ of the Poisson kernel $H_{D_0}(\cdot, P)$ in the domain D_0. In the limit $N \to \infty$, we will have a collection of continuous paths $\{\gamma\}$, which are the complex (*i.e.* the two-dimensional) Brownian motions (BM) running from O to P in D_0. The total mass of them is given by $C(D_0; O, P)$. The measure on the continuous path space of BMs is called the Wiener measure.

Loop-erased RW (LERW) model

If the lattice path $\omega \in \Omega_N(D_0; O, P)$ has the vertices such that $\omega(i) = \omega(j), i < j$, then the lattice path is said to be self-intersecting or having loops. From such a lattice path, we can obtain a non-self-intersecting sub-lattice-path $\widehat{\omega} = (\widehat{\omega}(0), \widehat{\omega}(1), \dots)$ by erasing loops chronologically: set $t_0 = 0, \widehat{\omega}(0) = \omega(t_0) = 0$, and for $m \geq 1$ let

$$t_m = \max\Big\{\ell > t_{m-1} : \omega(\ell) = \omega(t_{m-1}+1)\Big\}, \quad \widehat{\omega}(m) = \omega(t_m) = \omega(t_{m-1}+1).$$

We write the set of all non-self-intersecting lattice paths in ND_0 running from $NO = 0$ to NP as $\Omega^0_N(D_0; O, P)$. Each element of this set can be obtained from lattice paths in $\Omega_N(D_0; O, P)$ by the above mentioned loop-erasing procedure. Note that the map from lattice paths in $\Omega_N(D_0; O, P)$ to lattice paths in $\Omega^0_N(D_0; O, P)$ is many-to-one. To each element of $\Omega^0_N(D_0; O, P)$, we put the sum of measures $\sum_\omega 4^{-|\omega|}$ of all RWs that are mapped to the loop-erased lattice path. The statistical ensemble of non-self-intersecting lattice paths weighted in this way is called the loop-erased random walk (LERW) model.

To take a continuum limit of LERW, we introduce an index $\nu > 0$ and set

$$\omega^{1/N}\left(\frac{i}{N^{1/\nu}}\right) = \frac{1}{N}\omega(i), \quad 0 \leq i \leq |\omega| \tag{1.3}$$

for each $\omega = (\omega(0), \dots, \omega(|\omega|)) \in \Omega^0_N(D_0; O, P), N \in \mathbb{N}$. The obtained path $\omega^{1/N}$ is a non-self-intersecting lattice path in D_0 starting from the origin $O = 0$ and arriving at $P = 2\sqrt{-1}$ at the $|\omega|/N^{1/\nu}$-th step, where the Euclidean length of each step is $1/N$. It is expected that, if we choose the value of ν suitably, we can obtain the statistical ensemble of continuous paths γ in D_0 running from O to P in the limit $N \to \infty$:

$$\gamma : (0, t_\gamma) \to D_0 \quad \text{continuous}, \qquad \lim_{t \downarrow 0} \gamma(t) = O, \quad \lim_{t \uparrow t_\gamma} \gamma(t) = P, \quad t_\gamma \in (0, \infty). \tag{1.4}$$

Here the 'time' when each continuous path γ arrives at the point P, $t_\gamma = \lim_{N\to\infty} |\omega|/N^{1/\nu}$ will be a random variable. We expect that the fractal dimension of the paths in the continuum limit is $d_{\mathrm{LERW}} = 1/\nu$, and γ is simple; $\gamma(t_1) \neq \gamma(t_2), 0 \leq t_1 < t_2 \leq t_\gamma$. The space of all continuous paths (1.4) obtained by the continuum limit of the LERWs is denoted by $\mathcal{K}_{\mathrm{LERW}}(D_0; O, P)$ and the measure on this space by $\overline{\mu}^{\mathrm{LERW}}_{(D_0;O,P)}$. By definition of LERW, the total mass is the same as that of the continuum limit of RWs (*i.e.* the complex BMs) and given by $C(D_0; O, P)$. Then

$$\overline{\mu}^{\mathrm{LERW}}_{(D_0;O,P)}(\,\cdot\,) = C(D_0; O, P)\mu^{\mathrm{LERW}}_{(D_0;O,P)}(\,\cdot\,) \tag{1.5}$$

will define the probability measure $\mu^{\mathrm{LERW}}_{(D_0;O,P)}$ having a support on $\mathcal{K}_{\mathrm{LERW}}(D_0; O, P)$.

Self-avoiding walk (SAW) model

As a subset of W_n^z, consider a set of all non-self-intersecting paths

$$W_{n,0}^z = \Big\{\omega \in W_n^z : \omega(i) \neq \omega(j) \text{ for any } 0 \leq i < j \leq n\Big\}.$$

By this definition $|W_{n,0}^z| < |W_n^z| = 4^n$. We can show that there is $2 < e^\beta < 3$ such that

$$|W_{n,0}^z| \simeq e^{\beta n}, \quad n \to \infty,$$

where $f(n) \simeq g(n), n \to \infty$ means $\log f(n) \sim \log g(n), n \to \infty$. (The value e^β is called the SAW connective constant, which depends on the lattice structure. The exact value for $\mathbf{S}$ is not know, but numerically evaluated as 2.638.) Then we consider the statistical ensemble of non-self-intersecting paths by giving the weight $e^{-\beta|\omega|}$ to each element ω having length $|\omega|$. This ensemble is called the self-avoiding walk (SAW) model. The partition function of SAW model, which is corresponding to (1.1) of RWs is given by

$$Z_N^{\text{SAW}}(D_0; O, P) = \sum_{\omega \in \Omega_N^0(D_0;O,P)} e^{-\beta|\omega|}.$$

It is conjectured that there is an index $b_{\text{SAW}} > 0$ such that

$$Z_N^{\text{SAW}}(D_0; O, P) \sim C^{\text{SAW}}(D_0; O, P) N^{-2b_{\text{SAW}}}, \quad N \to \infty. \tag{1.6}$$

(Note that for the RWs and the LERWs, we have seen from (1.2) that $b_{\text{RW}} = b_{\text{LERW}} = 1$.) The continuum limits γ of SAWs should be simple as those of LERWs, but the fractal dimension d_{SAW} will be different from d_{LERW}, since the measures are different from each other. We expect that a measure of the continuous simple paths $\overline{\mu}_{(D_0;O,P)}^{\text{SAW}}(\,\cdot\,)$ is obtained by taking the scaling limit of SAW model and the probability measure $\mu_{(D_0;O,P)}^{\text{SAW}}(\,\cdot\,)$ will be given by

$$\overline{\mu}_{(D_0;O,P)}^{\text{SAW}}(\,\cdot\,) = C^{\text{SAW}}(D_0; O, P)\mu_{(D_0;O,P)}^{\text{SAW}}(\,\cdot\,).$$

Critical percolation model

Here we put the triangular lattice on $\mathbb{C}$ instead of the square lattice; put $\tau = \exp(2\pi\sqrt{-1}/3)$ and set $\mathbf{T} = \{z_0 + (i + j\tau)\sqrt{3}a : i, j \in \mathbb{Z}\}$, where $a = 2/3$ and $z_0 = a\sqrt{-1}$. Then the planar dual lattice of $\mathbf{T}$ is given by the honeycomb lattice $\mathbf{H}$ with the lattice spacing a, which includes the origin O and the point $NP = 2N\sqrt{-1}, N \in \mathbb{N}$. At each vertex $z \in \mathbf{T}$ put a random variable $\eta(z) \in \{0, 1\}$ following the Bernoulli measure $\nu_p, 0 \leq p \leq 1$: $\nu_p(\eta(z) = 1) = p, \nu_p(\eta(z) = 0) = 1 - p$. That is, each $\eta(z)$ takes 1 with probability p and 0 with probability $1 - p$ independently of other variables $\eta(z'), z' \neq z$.

If we regard the vertices with $\eta(z) = 1$ as wet sites and those with $\eta(z) = 0$ dry sites, a configuration $\{\eta(z) : z \in \mathbf{T}\}$ can be considered to represent how water percolates on a surface of some material. The set of wet vertices, which can be connected by lattice paths with a chosen vertex, is called a percolation cluster including the chosen vertex. It is known that if $p \leq 1/2$ the percolation cluster including a specified vertex, say, the origin O is bounded with probability one, and if $p > 1/2$ the probability that the percolation cluster

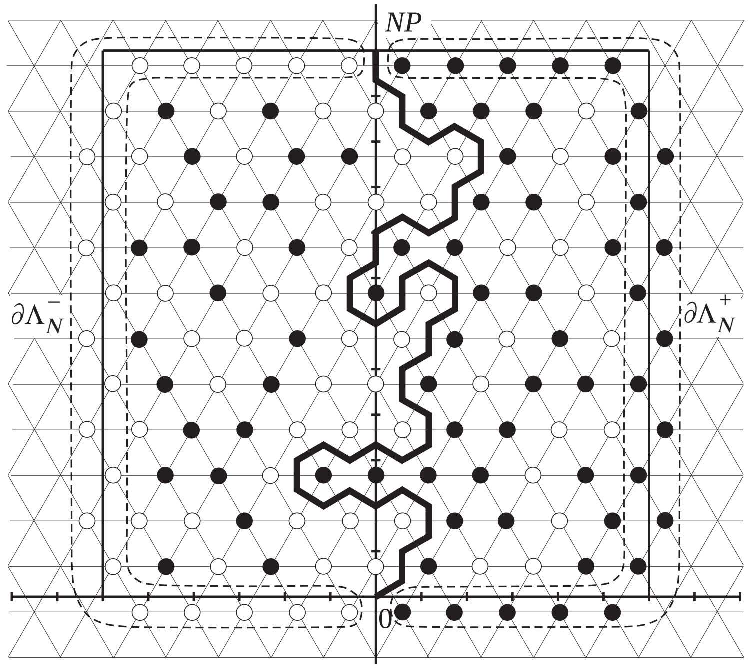

Figure 1: The percolation model on $\mathbf{T}$ and the percolation exploration process on $\mathbf{H}$. Wet vertices ($\eta(z) = 1$) are indicated by black dots and dray ones ($\eta(z) = 0$) by white dots.

including O is unbounded is positive. We call $p_c = 1/2$ the critical value and we say that there occurs a phase transition at $p_c = 1/2$. This model is called the percolation model. (The critical value p_c depends on lattice structure. For $\mathbf{T}$, $p_c = 1/2$, which is the reason why here we consider the model on $\mathbf{T}$.) Since the measure is Bernoulli, the total mass of measure (the partition function) equal to one independently of the size N of domain, which implies the index $b_{\text{per}} = 0$.

From now on, we consider the critical percolation model by setting $p = p_c = 1/2$. For a fixed $N \in \mathbb{N}$, let $\mathbf{T} \cap ND_0 = \Lambda_N$. Figure 1 shows the case with $N = 6$. As shown in this figure, let $\partial\Lambda_N^+$ be the set of vertices in $\mathbf{T}$ such that they are in the neighborhood of the boundary of ND_0 and located on the right of the straight line connecting O and P, and let $\partial\Lambda_N^-$ be the set of those vertices located on the left of the line. Then we assume that $\eta(z) = 1, \forall z \in \partial\Lambda_N^+$ and $\eta(z) = 0, \forall z \in \partial\Lambda_N^-$, which is called the Dobrushin boundary condition. The variables on other vertices inside of Λ_N, ${}^0\Lambda_N = \Lambda_N \cap (\partial\Lambda_N^+)^c \cap (\partial\Lambda_N^-)^c$, are randomly distributed following $\nu_{1/2}$.

For any realization of random configuration $\eta \in \{0,1\}^{{}^0\Lambda_N}$, there is a unique lattice path ω on $\mathbf{H} \cap N\overline{D}_0$ running from the origin O to NP such that it has all wet vertices on one side and all dry vertices on the other side in the neighborhood of the lattice path. This is called the percolation exploration process (see Fig.1).

If we choose an index $\nu > 0$ suitably and set (1.3), we will obtain a continuous path (1.4) with the fractal dimension $d_{\text{per}} = 1/\nu$ in the scaling limit $N \to \infty$. The continuous path γ is now not simple. The probability measure of the continuum limits of percolation exploration processes $\{\gamma\}$ is denoted by $\mu^{\text{per}}_{(D_0;O,P)}(\,\cdot\,)$.

Critical Ising model

Let $\overline{\Lambda}_N = \Lambda_N \cup \partial\Lambda_N^+ \cup \partial\Lambda_N^-$. For each vertex $z \in \overline{\Lambda}_N$ put a variable $\sigma(z) \in \{-1, 1\}$, which we call a spin variable. We assume the Dobrushin boundary condition: $\sigma(z) = \pm 1$ for $z \in \partial\Lambda_N^\pm$. Inside of the region ${}^0\Lambda_N$, spin variables are randomly distributed. The function $\sigma \in \{-1, 1\}^{\overline{\Lambda}_N} \to \mathbb{R}$ given by

$$E(\sigma) = -\frac{1}{2} \sum_{z,z' \in \overline{\Lambda}_N : |z-z'| = \sqrt{3}a} \sigma(z)\sigma(z')$$

is called the energy. Spin configurations are distributed with the Gibbs measure with parameter $\beta > 0$,

$$\pi_{N,\beta}(\sigma) = \frac{e^{-\beta E(\sigma)}}{Z_{N,\beta}}, \quad Z_{N,\beta} = \sum_{\sigma \in \{-1,1\}^{\overline{\Lambda}_N}} e^{-\beta E(\sigma)}$$

and this spin system is called the Ising model with the inverse temperature β. This is a model representing magnetization of some materials.

For each spin configuration, we can define the Ising exploration process on $\mathbf{H} \cap N\overline{D}_0$ in the similar way to the percolation exploration process. Note that in this case the lattice path of the Ising exploration process is sandwiched by sites with $\sigma(z) = 1$ and those with $\sigma(z) = -1$, while the lattice path of the percolation exploration process is done by wet sites and dry sites. In particular, we consider the Ising exploration process with the Gibbs measure at the critical value of parameter, which is known to be $\beta_c = (\log 3)/4$ for $\mathbf{T}$. For this critical Ising exploration processes, we can take the scaling limit and obtain the continuous paths $\{\gamma\}$ with the fractal dimension d_{Ising}. The continuous paths are expected to be simple. The measure of the paths is denoted by $\overline{\mu}^{\text{Ising}}_{(D_0;O,P)}(\,\cdot\,)$.

1.2 Conformal invariance and domain Markov property

If f is holomorphic in $D_0 \subset \mathbb{C}$ and $f'(z) \neq 0, \forall z \in D_0$, we call

$$f : D_0 \quad \to \quad f(D_0) \tag{1.7}$$

a conformal transformation. Assume that by f the points O, P on the boundary ∂D_0 are mapped to the points $f(O), f(P)$ on $\partial f(D_0)$. The measures of continuum paths $\{\gamma\}$ obtained in the scaling limits of the statistical mechanics models on planar lattices discussed in Section 1.1

$$\overline{\mu}_{(D_0;O,P)}(\,\cdot\,) = C(D_0; O, P)\mu_{(D_0;O,P)}(\,\cdot\,) \tag{1.8}$$

are expected to have the following two properties.

Conformal covariance and conformal invariance

There is an index b such that for any conformal transformation (1.7), the equality

$$f \circ \overline{\mu}_{(D_0;O,P)}(\,\cdot\,) = |f'(O)|^b |f'(P)|^b \overline{\mu}_{(f(D_0);f(O),f(P))}(\,\cdot\,) \tag{1.9}$$

is established. This equality should be compared with the asymptotic behavior of partition functions of statistical mechanics models shown by, for example, (1.6). The scaling (1.3) corresponds to performing the conformal transformation by a simple dilatation $f(z) = z/N, N \in \mathbb{N}$ and in this case the factor in RHS of (1.9) becomes $|f'(O)|^b |f'(P)|^b = N^{-2b}$. Then the index b in (1.9) is identified with the index showing asymptotics of the partition function in the limit $N \to \infty$. By the form of (1.9), b is called the boundary scaling exponent. Equality (1.9) implies the conformal covariance of the total mass of measure

$$C(D_0; O, P) = |f'(O)|^b |f'(P)|^b C(f(D_0); f(O), f(P))$$

and the conformal invariance of the probability measure

$$\mu_{(D_0;O,P)}(\,\cdot\,) = \mu_{(f(D_0);f(O),f(P))}(\,\cdot\,). \tag{1.10}$$

Domain Markov Property

With $\mu_{(D_0;O,P)}$, suppose that we observe an initial segment of the path up to a time t denoted by $\gamma(0,t]$ for an arbitrary $t \in (0, t_\gamma)$. Then the conditional distribution of the remaining part of the path is the same as the distribution of path in the domain $D_0 \setminus \gamma(0,t]$, which starts from $\gamma(t)$ and arrives at $\gamma(t_\gamma) = P$:

$$\mu_{(D_0;O,P)}\Big(\,\cdot\,\Big|\gamma(0,t]\Big) = \mu_{(D_0 \setminus \gamma(0,t];\gamma(t),P)}(\,\cdot\,).$$

This is called the domain Markov property.

1.3 Restriction property and locality property

In some special cases the measure (1.8) can have the following properties in addition to the conformal covariance/invariance and the domain Markov property.

Restriction property

Consider a simply connected subdomain $D_1 \subset D_0$ such that $O, P \in \partial D_0 \cap \partial D_1$. (A domain D is said to be simply connected if $\widehat{\mathbb{C}} \setminus D$ is a connected subset of the Riemann sphere $\widehat{\mathbb{C}}$.) We consider the LERW in this subdomain D_1 and perform the scaling limit as explained in Section 1.1 to define the measure $\overline{\mu}^{\text{LERW}}_{(D_1;O,P)}$ for continuous paths. Since the domain D_1 is a subset of D_0 and lattice paths of RWs are restricted by narrowing the region where RW can run, the measures added to each LERW from RWs in D_1 in the loop-erasing procedure will be decreased. Then in general for the Radon-Nikodym derivative

$$\frac{d\overline{\mu}^{\text{LERW}}_{(D_1;O,P)}}{d\overline{\mu}^{\text{LERW}}_{(D_0;O,P)}} < 1, \quad D_1 \subset D_0, \quad D_1 \neq D_0.$$

For the continuum limit of the SAW model, however, the following equality will hold,

$$\frac{d\overline{\mu}^{\text{SAW}}_{(D_1;O,P)}}{d\overline{\mu}^{\text{SAW}}_{(D_0;O,P)}} = \mathbf{1}\{\gamma(0,t_\gamma) \subset D_1\}, \quad D_1 \subset D_0, \tag{1.11}$$

where $\mathbf{1}\{\omega\}$ is an indicator function of the event ω. The equality (1.11) is called the restriction property.

Locality property

Since the random variables η's in the percolation model are distributed following the Bernoulli measure, the behavior of the percolation exploration process depends only on the η-configurations on the vertices which are adjacent to the lattice path of the process. Then the probability measure μ^{per} of the continuous paths obtained in the scaling limit is expected to have the following property. For a simply connected subdomain $D_1 \subset D_0$ with $O, P \in \partial D_0 \cap \partial D_1$

$$\mu^{\mathrm{per}}_{(D_1;O,P)}(\gamma(0,t]) = \mu^{\mathrm{per}}_{(D_0;O,P)}(\gamma(0,t])\mathbf{1}\{\gamma(0,t) \subset D_1\}, \quad \forall t \in (0, t_\gamma). \tag{1.12}$$

This is called the locality property. We can not expect this property for $\overline{\mu}^{\mathrm{Ising}}$.

Remark that the restriction property (1.11) is the property for the entire path $\gamma(0, t_\gamma)$, but the locality property (1.12) should hold for any initial segment $\gamma(0,t], t \in (0, t_\gamma)$. Then the locality property is stronger than the restriction property.

2 Stochastic Loewner Evolution (SLE)

2.1 Riemann mapping theorem

Let D, D' be simply connected domains in $\mathbb{C}$ and assume that $D, D' \neq \mathbb{C}$, $z, w \in \partial D$, and $z', w' \in \partial D'$. By the Riemann mapping theorem, we can conclude that there is a one-parameter family of

$$\text{conformal transformations} \quad f : D \to D', \quad \text{s.t.} \quad f(z) = z', \quad f(w) = w'.$$

If we assume the condition $|f'(w)| = 1$, then the conformal transformation is uniquely determined.

Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$. From the above result, if a conformally invariant probability measure $\mu_{(\mathbb{H};0,\infty)}$ is given, we can obtain the probability measure $\mu_{(D';z',w')}$ for any simply connected domain $D' \subset \mathbb{C}, D' \neq \mathbb{C}$ with $z', w' \in \partial D'$ by the conformal transformation $f : D \equiv \mathbb{H} \to D'$. Moreover, if $w' = \infty \in \partial D'$ for the domain $D' \subset \mathbb{H}$ and $f(\infty) = \infty$, the assumption $|f'(w)| = 1$ simplifies the conformal covariance (1.9) of the measure to be $f \circ \overline{\mu}_{(\mathbb{H};x,\infty)}(\,\cdot\,) = |f'(x)|^b \overline{\mu}_{(D';z',\infty)}(\,\cdot\,), x \in \mathbb{R} = \partial\mathbb{H} \setminus \{\infty\}, f(x) = z' \in \partial D' \setminus \{\infty\}$.

2.2 Loewner differential equation

Suppose that $\gamma : (0,\infty) \to \mathbb{H}$ is a simple path from $\lim_{t\downarrow 0} \gamma(t) = U_0 \in \mathbb{R}$ to $\lim_{t\uparrow\infty} \gamma(t) = \infty$. Then for each $t \in (0,\infty)$ there is a unique conformal transformation $f : \mathbb{H} \setminus \gamma(0,t] \to \mathbb{H}$ such that $\lim_{z\to\infty} f'(z) = 1$, which is denoted by g_t. It has the expansion

$$g_t(z) = z + \frac{a(t)}{z} + \mathcal{O}(|z|^{-2}), \quad z \to \infty, \tag{2.1}$$

where $a(t) > 0$ is the half-plane capacity of the path $\gamma(0,t]$. The conformal transformation $g_t(z)$ solves the following differential equation

$$\frac{d}{dt} g_t(z) = \frac{da(t)/dt}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{2.2}$$

which is called the (chordal) Loewner equation. Here $U_t = g_t(\gamma(t)) \in \mathbb{R}$ in the sense

$$\lim_{z \to \gamma(t), z \in \mathbb{H} \setminus \gamma(0,t]} g_t(z) = U_t, \tag{2.3}$$

and $U_t : [0, \infty) \to \mathbb{R}$ is continuous. The family of conformal transformations parameterized by time $(g_t)_{t \geq 0}$ is called the Loewner chain.

In the above, given a continuous path $\gamma(0,t]$ and a continuous function U_t, $t \in (0, \infty)$, the equation (2.2) is introduced. We will change the situation. Given a strictly increasing C^1 function $a : [0, \infty) \to [0, \infty)$ with $a(0) = 0$ and a continuous function $U_t : [0, \infty) \to \mathbb{R}$, we will consider the Loewner equation (2.2). For each $z \in \mathbb{H}$, the solution of the equation $g_t(z)$ exists up to time $T_z = \sup\{t > 0 : g_t(z) - U_t \neq 0\}$. It can be shown that for fixed t, g_t is the conformal transformation of $H_t \equiv \{z \in \mathbb{H} : T_z > t\}$ onto $\mathbb{H}$ having the expansion (2.1) at infinity.

Corresponding to (2.3), the trace of Loewner chain is defined by

$$\gamma(t) \equiv \lim_{z \to 0, z \in \mathbb{H}} g_t^{-1}(z + U_t). \tag{2.4}$$

If the limit does not exist, let $\gamma(t)$ denote the set of all limit points. We say that the trace of Loewner chain is a continuous path, if the limit (2.4) exists for every $t \in [0, \infty)$ and $\gamma(t)$ is a continuous function of t.

2.3 SLE$_\kappa$

Now we put U_t be an appropriate stochastic process and consider the statistical ensemble of the continuous paths $\{\gamma\}$ which give the traces of the Loewner chains (2.4). We parameterize the path so that $a(t) = at, a > 0$. As the measure of the paths $\{\gamma\}$, we want to construct $\overline{\mu}_{(\mathbb{H};0,\infty)}$ so that it can describe the measure of continuous paths obtained by the scaling limit of statistical mechanics model discussed in Section 1. The condition that $\overline{\mu}_{(\mathbb{H};0,\infty)}$ is conformally covariant and has the domain Markov property requires U_t be continuous with stationary and independent increment. It implies that U_t is a one-dimensional Brownian motion (BM). (Here we are not interested in drift.)

In this setting, we can choose the variance of BM and the parameter a. Since the "speed of the growth" of the path γ is irrelevant for statistics of γ (*i.e.* we can perform time change), there is only one free parameter. Following the original paper by Schramm [6], we set $a = 2$ and a single positive parameter κ is used for the variance of BM. The (chordal) stochastic Loewner evolution (Schramm-Loewner evolution) with parameter κ is the solution of

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z, \tag{2.5}$$

where B_t is a standard (*i.e.* the variance 1) one-dimensional BM with $B_0 = 0$. (Note that $\sqrt{\kappa}B_t$ has the same distribution as $B_{\kappa t}$.) If the trace of g_t is a random continuous path γ, then γ is called the SLE$_\kappa$ path.

By the above reasoning, any measure on the continuous path space, which has the conformal covariance/invariance and the domain Markov property, is given by an appropriate conformal transformation of an element of the one-parameter family of measures for SLE$_\kappa$ paths, which is denoted by $\{\overline{\mu}^{\kappa}_{(\mathbb{H};0,\infty)}\}_{\kappa>0}$.

2.4 Basic properties of SLE$_\kappa$

In the present talk, I will explain the following basic properties of SLE$_\kappa$. The statements hold with probability one.

(1) SLE$_\kappa$ trace is a continuous path for $\kappa > 0$ [5, 4].

(2) There are three phases of SLE$_\kappa$ paths [2, 3]. (a) If $\kappa \leq 4$, SLE$_\kappa$ paths are simple and $\gamma(0,\infty) \in \mathbb{H}$. (b) If $4 < \kappa < 8$, SLE$_\kappa$ paths are self-intersecting and $\gamma(0,\infty) \cap \mathbb{R} \neq \emptyset$, but for each $z \in \mathbb{H}$, $\mathbf{P}[z \in \gamma(0,\infty)] = 0$. (c) If $\kappa \geq 8$, SLE$_\kappa$ paths are self-intersecting and $\gamma[0,\infty) = \overline{\mathbb{H}}$, *i.e.* γ is plane-filling. See Figure 2.

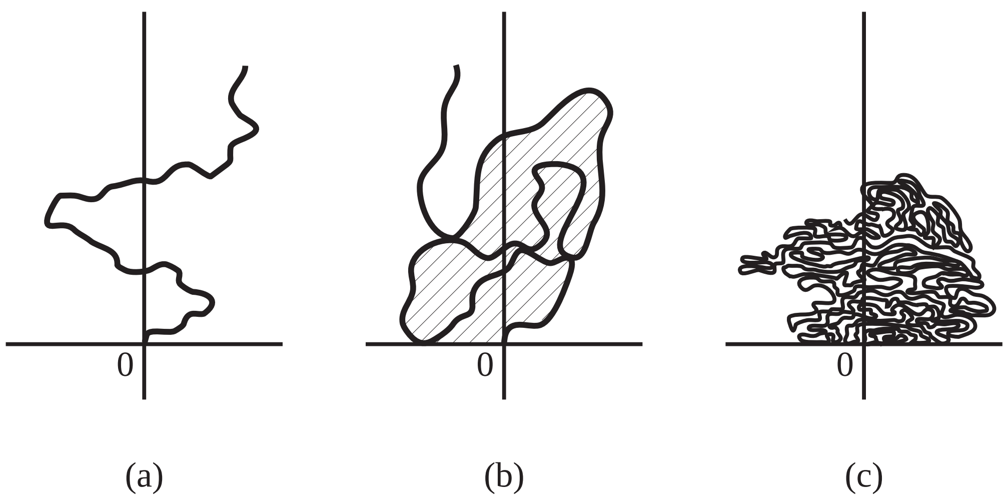

Figure 2: The three phases of SLE$_\kappa$ paths.

(3) The Hausdorff dimension of SLE$_\kappa$ path is given by [1]

$$d(\kappa) = \min\left\{1 + \frac{\kappa}{8}, 2\right\}, \quad \kappa > 0. \tag{2.6}$$

(4) SLE$_\kappa$ measure $\overline{\mu}^{\kappa}_{(\mathbb{H};0,\infty)}$ has the locality property if and only if $\kappa = 6$ [2, 3].

(5) The boundary scaling exponent b of SLE$_\kappa$ path is given by [2, 3]

$$b(\kappa) = \frac{6-\kappa}{2\kappa}, \quad \kappa > 0. \tag{2.7}$$

(6) SLE$_\kappa$ measure $\overline{\mu}^{\kappa}_{(\mathbb{H};0,\infty)}$ has the restriction property if and only if $\kappa = \dfrac{8}{3}$ [2, 3].

3 Correspondence

If conformal invariance of $\mu^{\rm per}_{(D_0;0,P)}$ and conformal covariance of $\overline{\mu}^{\rm SAW}_{(D_0;O,P)}$ are proved, the former will be identified with $\mu^{\kappa=6}_{(\mathbb{H};0,\infty)}$ and latter with $\overline{\mu}^{\kappa=8/3}_{(\mathbb{H};0,\infty)}$ (mod. conformal transformation). The former was proved by Smirnov [7], but the latter is open. In addition to that, the following were proved: $\overline{\mu}^{\rm LERW}_{(D_0;0,P)} = \overline{\mu}^{\kappa=2}_{(\mathbb{H};0,\infty)}$ [6, 4] and $\overline{\mu}^{\rm Ising}_{(D_0;0,P)} = \overline{\mu}^{\kappa=3}_{(\mathbb{H};0,\infty)}$ [8]. The both equalities hold if we perform appropriate conformal transformation $f : \mathbb{H} \to D_0$.

Combining with the results (2.6) and (2.7) on $d(\kappa)$ and $b(\kappa)$, the indices (critical exponents) of the statistical mechanics models on the planar lattices will be determined as follows:

$$d_{\rm LERW} = d(2) = \frac{5}{4}, \quad d_{\rm SAW} = d(8/3) = \frac{4}{3}, \quad d_{\rm Ising} = d(3) = \frac{11}{8}, \quad d_{\rm per} = d(6) = \frac{7}{4},$$
$$b_{\rm LERW} = b(2) = 1, \quad b_{\rm SAW} = b(8/3) = \frac{5}{8}, \quad b_{\rm Ising} = b(3) = \frac{1}{2}, \quad b_{\rm per} = b(6) = 0.$$

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