Curvatures of Carathéodory pseudo-volume forms and its applications *

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1 Introduction

In this talk, we report our trial up to now clarifying explicit relations between the measure hyperbolicities and the positivities of the canonical bundle. For this purpose, our method is to investigatie the curvatures of the pseudo-volume forms defining the measure hyperbolicities. We particularly consider the Carathéodory measure hyperbolic case at this time.

The concepts of measure hyperbolicities appear when the Schwarz lemma is generalized to a higher dimensional case if it is regarded as the volume decreasing property. For instance, Kobayashi measure hyperbolicity and its dual version, Carathéodory measure hyperbolicity, are two of them. They are defined by some positivity conditions of some intrinsic pseudo-volume forms (for example, Carathéodory measure hyperbolicity is defined by the Carathéodory pseudo-volume form, and so on). By their definition, it can be expected that the measure hyperbolic manifold looks like a manifold with a metric whose Ricci curvature is negative. Hence it is very likely that the measure hyperbolicities lead to the positivities of the canonical bundle.

On the other hand, the positivities of the canonical bundle are closely related to the curvatures of the (possibly singular) pseudo-volume form. Here we note that the inverse of the pseudo-volume form can be regarded as a singular Hermitian metric on the canonical bundle.

Therefore to gain the positivity of the canonical bundle, it is the most natural and direct way to study the curvature of these pseudo-volume forms defining the measure hyperbolicities.

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Among the precedent results about a relation between the measure hyperbolicities and the positivities of the canonical bundle, the following is very famous:

Theorem 1.1. (*Kobayashi-Ochiai* [17]) *Every projective manifold of general type is Kobayashi measure hyperbolic.*

Moreover the converse of this theorem is conjectured by S. Kobayashi.

Conjecture 1. *Every Kobayashi measure hyperbolic projective manifold is of general type.*

This conjecture is shown to be true in the 2-dimensional case by using the Kodaira-Enriques classification of complex projective surfaces ([14], [19]).

As regards the curvatures of the intrinsic objects, the metric case is already known.

Theorem 1.2. ([22], [4], [21]) *The holomorphic sectional curvature of the Kobayashi* (*resp. Carathéodory*) *pseudo metric is greater* (*resp. less*) *than or equal to* -1.

Hence one of our aims is to obtain a volume version of this theorem, and we discuss the Carathéodory pseudo-volume form version in Section 2. Then we have the following:

Theorem 1.3. ([15]) *The curvature of the Carathéodory pseudo-volume form* v_X^C *of a complex manifold* X *is bounded above by* -1, *and its curvature current is strictly positive outside its zero locus.*

Thanks to Theorem 1.3, we can see that Carathéodory measure hyperbolicity implies the positivity of the canonical bundle K_X of a compact complex manifold X.

Corollary 1.1. ([15]) *Let* X *be a compact normal complex space and* $\tilde{X}$ *its universal covering space.*

Then we have

$$\mathrm{vol}(K_X) \geq \frac{n!(n+1)^n}{(4\pi)^n}\mu_{\tilde{X}}^C(X),$$

where $\mu_{\tilde{X}}^C$ *is the Carathéodory measure on* $\tilde{X}$ *and considered as on* X. *Especially, if* $\tilde{X}$ *is Carathéodory measure hyperbolic, then* X *is of general type.*

If X *is smooth and* $\tilde{X}$ *is strongly Carathéodory measure hyperbolic, then* X *is projective algebraic with the ample canonical bundle.*

To investigate the positivities of the canonical bundle in the analytic fashion, one of standard ways is an analysis of the Bergman kernel. Moreover we should emphasize that the Carathéodory pseudo-volume form and the Bergman kernel form are intrinsic, that is, they depend only on the complex structure of X. Thanks to Theorem 1.3, we can also see that Carathéodory measure hyperbolicity, i.e., the positivity of the Carathéodory pseudo-volume form implies both the positivities of the Bergman kernel form and metric.

Theorem 1.4. *Assume that X is a Carathéodory measure hyperbolic manifold with a smooth compact quotient. Then the Bergman kernel form is positive and the Bergman metric is strictly positive on the open set where the Carathéodory pseudo-volume form v_X^C is positive.*

The restricted version of the above considerations is also observed in Section 3, and the analogous statement is shown to hold.

Finally we mainly investigate Carathéodory measure hyperbolicity so far. It is because Carathéodory measure hyperbolicity and the pseudo-volume form are simpler to treat and these structures are more rich than Kobayashi's ones. Regarding the curvature of the Kobayashi pseudo-volume form, the following is conjectured by H. Tsuji.

Conjecture 2. (*H. Tsuji*) *The curvature current of the Kobayashi pseudo-volume form is positive over a projective manifold with the ample canonical bundle.*

2 Carathéodory measure hyperbolicity and positivity of canonical bundle

Let X be a connected paracompact complex manifold in this paper unless further notice. At first, we recall the definitions of the Carathéodory pseudo-volume form and measure:

Definition 2.1. ([22]) The Carathéodory pseudo-volume form v_X^C on a complex manifold X is defined by setting

$$v_X^C := \sup\{f^*v_1\,;\, f \in \mathrm{Hol}(X, \mathbb{B}^n)\},$$

where $v_1 = 2^n(1-|z|^2)^{-(n+1)} \bigwedge_{\alpha=1}^n dz^\alpha \wedge d\overline{z}^\alpha$ is the Poincaré volume form on the n-dimensional complex ball $\mathbb{B}^n$ and $\mathrm{Hol}(X, \mathbb{B}^n)$ is the space consisting of all holomorphic mappings from X to $\mathbb{B}^n$.

When X is a normal complex space, the Carathéodory measure on X is defined as follows: for each Borel set $B \subset X$, we put

$$\mu_X^C(B) := \sup \Bigg\{ \sum_{i=1}^{\infty} \mu_1(f_i(B_i))\,;\ f_i \in \mathrm{Hol}(X, \mathbb{B}^n)\ (i \in \mathbb{N}) \text{ and}$$
$$(B_i)_{i\in\mathbb{N}} : \text{mutually disjoint Borel sets of } X \text{ s.t. } \bigcup_{i=1}^{\infty} B_i = B \Bigg\},$$

where μ_1 denotes the measure on $\mathbb{B}^n$ with v_1 as its density.

The definition of Carathéodory measure hyperbolicity is given as follows:

Definition 2.2. *A normal complex space X is said to be Carathéodory measure hyperbolic if μ_X^C is nontrivial.*

A complex manifold X is said to be strongly Carathéodory measure hyperbolic if v_X^C is positive everywhere.

When X is normal, we denote by X_{reg} the regular set of X. Then it follows that $\mu_X^C = \mathbf{1}_{X_{\mathrm{reg}}} v_{X_{\mathrm{reg}}}^C$ by the extension of bounded holomorphic functions. Here $\mathbf{1}_{X_{\mathrm{reg}}}$ is the characteristic function of X_{reg}.

One of important properties of the Carathéodory pseudo-volume form is the volume decreasing property which is a generalization of the Schwarz lemma:

Theorem 2.1. *For every holomorphic map $f : X \to Y$ between n-dimensional complex manifolds, we have*

$$f^*\mu_Y^C \le \mu_X^C.$$

Especially the Carathéodory pseudo-volume form and measure are biholomorphically invariant.

Moreover since $\mathbb{B}^n$ is homogeneous, we can express the definition formula of v_X^C by the Ascoli-Arzelà theorem as follows: for any $x \in X$, there exists $f_x \in \mathrm{Hol}(X, \mathbb{B}^n)$ with $f_x(x) = o$ such that

$$(v_X^C)_x = \sup\{(f^*v_1)_x \,:\, f \in \mathrm{Hol}(X, \mathbb{B}^n),\ f(x) = o\} = (f_x^* v_1)_x.$$

Therefore the pseudo-volume form is continuous on X. Here we set a closed analytic subset $\mathrm{Z}(v_X^C) := \{x \in X\,;\ (v_X^C)_x = 0\}$ of X.

Next we define the curvature for a pseudo-volume form v on X such that its curvature current $\Theta_{v^{-1}} = \sqrt{-1}\partial\overline{\partial}\log v$ is positive. Here we regard v^{-1} as a singular Hermitian metric on the canonical bundle K_X of X. Then it can be seen that $\Theta_{v^{-1}}$ becomes the $(1,1)$-form with the Radon measure coefficients.

Definition 2.3. ([15]) For a pseudo-volume form v on X such that its curvature current $\Theta_{v^{-1}}$ is positive, we define the curvature K_v by setting

$$K_v := -\frac{2^n}{(n+1)^n n!}\frac{(\sqrt{-1}\partial\overline{\partial}\log v)^n_{\mathrm{ac}}}{v},$$

where "ac" means the absolutely continuous part of the Lebesgue decomposition of $\sqrt{-1}\partial\overline{\partial}\log v$ with respect to the Lebesgue measure.

Note that we can easily observe that $\Theta_{(v_X^C)^{-1}} = \sqrt{-1}\partial\overline{\partial}\log v_X^C$ is positive, and thus that the curvature $K_{v_X^C}$ of v_X^C makes sense. In this formulation, we rewrite the statement of Theorem 1.3.

Theorem 2.2. *We have*

$$K_{v_X^C} \leq -1$$

and

$$\Theta_{(v_X^C)^{-1}} > 0 \quad \text{on } X \setminus \mathrm{Z}(v_X^C).$$

This theorem is proved as follows: the first statement is obtained as we apply the next key lemma to

$$T_k := \sqrt{-1}\partial\overline{\partial}\log v_k^C$$

for the approximating sequence

$$v_k^C := \sup_{i=1,\cdots,k} f_{x_i}^* v_1, \ k \in \mathbb{N}$$

of v_X^C, where $\{x_k\}_{k\in\mathbb{N}} \subset X$ is a dense subset of X.

Lemma 2.1 ([2])**.** *Let $(T_k)_{k\in\mathbb{N}}$, T are positive $(1,1)$-currents on X. If $T_k \xrightarrow{k\to\infty} T$, we have $(T_{\mathrm{ac}})^n \geq \liminf_{k\to\infty}(T_k)^n_{\mathrm{ac}}$.*

Moreover we use the simple fact that $\sqrt{-1}\partial\overline{\partial}\log f_x^* v_1 \geq f_x^*\sqrt{-1}\partial\overline{\partial}\log v_1$.

The second statement follows because $\sqrt{-1}\partial\overline{\partial}\log f_x^* v_1$ is uniformly strictly positive with respect to $x \in K$ for any compact $K \subset X \setminus \mathrm{Z}(v_X^C)$. This is because v_X^C is continuous.

Second, we prove Corollary 1.1. First of all, we recall the definition of the volume of the canonical bundle K_X of a compact normal complex space X:

$$\mathrm{vol}(K_X) := \limsup_{m\to\infty} n!\frac{h^0(mK_X)}{m^n}.$$

As regards the first part of this corollary, the singular case reduces to the smooth case by the resolution of singularities of X and the normality of X. We combine the first part of Theorem 1.3 with a formula for the volume of a line bundle L by Boucksom and Popovici:

Theorem 2.3. ([2], [20]) *Let X be a compact complex manifold and L a holomorphic line bundle over X. Then we have*

$$\mathrm{vol}(L) = \sup\left\{\int_X T_{\mathrm{ac}}^n \,;\, T \in c_1(L) \text{ is a positive } (1,1)\text{-current}\right\}.$$

We will apply this result with $L = K_X$. Note that the Carathéodory pseudo-volume form $v_{\tilde{X}}^C$ of the universal covering $\tilde{X}$ on X is biholomorphically invariant. Thus, we can regard $(v_{\tilde{X}}^C)^{-1}$ as a singular Hermitian metric on K_X. Hence we can take $T = (2\pi)^{-1}\Theta_{(v_{\tilde{X}}^C)^{-1}} \in c_1(K_X)$ in Theorem 2.3. From first part of Theorem 1.3, we conclude that

$$\mathrm{vol}(K_X) \geq \int_X \left(\frac{1}{2\pi}\sqrt{-1}\partial\overline{\partial}\log v_{\tilde{X}}^C\right)_{\mathrm{ac}}^n \geq \frac{n!(n+1)^n}{(4\pi)^n}\mathrm{vol}_{v_{\tilde{X}}^C}(X).$$

To prove the second part of this corollary, we apply Richberg's regularization technique ([6]) to the continuous strictly plurisubharmonic function $\log v_{\tilde{X}}^C$. We use this to regularize the singular Hermitian metric $(v_{\tilde{X}}^C)^{-1}$ while keeping the strict positivity of the curvature current. Hence we get a smooth Hermitian metric on K_X that has the strictly positive curvature form on X. By Kodaira's embedding theorem, X turns out to be a projective algebraic manifold with ample K_X.

Finally, we prove Theorem 1.4. We denote by

$$(\alpha, \beta) := \sqrt{-1}^{n^2}\int_X \alpha \wedge \overline{\beta}$$

the Hermitian inner product of two $(n,0)$-forms α, β. Recall the definitions of the Bergman kernel and metric:

Definition 2.4. Let $\{\alpha_i\}_{i=0}^{\infty}$ be a complete orthonormal system of the Hilbert space $L^2(\Omega_X^n)$ consisting of the holomorphic n-form α satisfying $\|\alpha\|^2 = (\alpha, \alpha) < \infty$. The Bergman kernel v_X^B on X is defined by

$$v_X^B := \sum_{i=0}^{\infty}\sqrt{-1}^{n^2}\alpha_i \wedge \overline{\alpha_i} = \sup_{\alpha \in L^2(\Omega_X^n),\, \|\alpha\|=1}\sqrt{-1}^{n^2}\alpha \wedge \overline{\alpha}.$$

The pseudo-metric associated to the curvature form $\Theta_{(v_X^B)^{-1}}$ of the singular Hermitian metric $(v_X^B)^{-1}$ as the fundamental $(1,1)$-form is called the Bergman pseudo-metric on X. When $\Theta_{(v_X^B)^{-1}}$ is nondegenerate on X, we call it the Bergman metric on X.

To prove Theorem 1.4, we start by using some L^2 estimate for the $\overline{\partial}$ operator in order to construct the suitable elements in $L^2(\Omega_X^n)$ by following Chen [5].

By the hypothesis, X covers a compact complex manifold Y and let $\pi : X \to Y$ be the covering map. We can consider v_X^C as defined on Y. We proceed to construct the weight function and the complete Kähler domain in X to set up the suitable L^2 space in order to use the L^2 estimate for the $\overline{\partial}$ operator.

First we construct the weight function. Let $\{x_i\}_{i\in\mathbb{N}} \subset X$ be a countable dense subset such that

$$X \setminus Z(v_X^C) = \bigcup_{i=1}^{\infty} \left(df_{x_i}^1 \wedge df_{x_i}^2 \wedge \cdots \wedge df_{x_i}^n \neq 0 \right)$$

for $f_{x_i} = (f_{x_i}^1, \ldots, f_{x_i}^n) \in \mathrm{Hol}(X, \mathbb{B}^n)$. Consider the smooth and bounded plurisubharmonic function

$$\phi := \log \left(1 + \sum_{i=1}^{\infty} \frac{1}{2^i} \|f_{x_i}\|^2 \right),$$

where $\|f_{x_i}\|^2 := \sum_{j=1}^{n} |f_{x_i}^j|^2 < 1$. In fact, it can be easily seen that

Lemma 2.2. *ϕ is strictly plurisubharmonic on $X \setminus Z(v_X^C)$.*

Second, we construct a complete Kähler domain in X. The hypothesis that X is Carathéodory measure hyperbolic implies that Y is of general type by Corollary 1.1. Hence Moishezon's theorem implies that there exists a projective manifold Y' obtained after finitely many blow-ups $\sigma_Y : Y' \to Y$ along smooth centers. Then we have a Galois covering $X' \xrightarrow{\pi'} Y'$ of Y' such that the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\sigma_X} & X \\ \pi' \downarrow & \circlearrowright & \downarrow \pi \\ Y' & \xrightarrow{\sigma_Y} & Y \end{array}$$

holds. By the biholomorphic invariance and continuity of the Carathéodory pseudo-volume form and the diagram, we have ${\sigma_X}^* v_X^C = v_{X'}^C$. We can denote $Z(v_{X'}^C) = E \cup \sigma_X^{-1}(Z(v_X^C))$ for some proper analytic subset E of X and the analytic subset $S = \sigma_X(E)$ of X satisfies

$$\sigma_X : X' \setminus Z(v_{X'}^C) \xrightarrow{\sim} X \setminus \left(S \cup Z(v_X^C) \right) \tag{1}$$

is a biholomorphism.

By this biholomorphism, the domain $D := X \setminus \big(S \cup Z(v_X^C)\big)$ is the desired complete Kähler domain in X since $X' \setminus Z_{X'}^C$ has a complete Kähler metric. This follows by a similar way to Demailly [9, p.471, Lemma 7.2] because Y' is compact Kähler.

To complete the proof, we use the next result by Chen in [5] applied to the complete Kähler manifold D:

Proposition 2.1. ([5]) *Let (M, ω_M) be a complete Kähler manifold. Suppose that there exists a bounded smooth strictly plurisubharmonic function ϕ on M. Then M possesses a Bergman kenel and metric.*

This proposition directly implies the existence of the Bergman kernel and metric on $D = X \setminus \big(S \cup Z(v_X^C)\big)$. In regard to the existence on $X \setminus Z(v_X^C)$, we observe that the proof of Proposition 2.1 turns out to be applicable to every point in $S \setminus Z(v_X^C)$ since the ϕ constructed above is strictly plurisubharmonic there. The existence of the Bergman kernel and of the Bergman metric on $X \setminus Z(v_X^C)$ follows.

3 Restricted Carathéodory measure and restricted volume of canonical bundle

In this section, we consider the restricted version of Corollary 1.1. The restricted version of the volume, so called the restricted volume, of a line bundle L over a compact complex manifold X is a useful tool for extending sections from a subvariety to an ambient space. And so various properties and applications of the notion are obtained (see [10], [18], [12]). Its precise definition are given by the following way: let L be a line bundle over a compact complex manifold X and Z its d-dimensional irreducible closed complex subspace. Denote by $\iota : Z \hookrightarrow X$ the inclusion map. We consider the spaces

$$H^0(X|Z, \mathcal{O}(mL)) := \mathrm{Im}[\iota^* : H^0(X, \mathcal{O}(mL)) \to H^0(Z, \mathcal{O}_Z(mL))]$$

for all $m \in \mathbb{N}$. Then the restricted volume $\mathrm{vol}_{X|Z}(L)$ of L to Z is defined as

$$\mathrm{vol}_{\mathrm{X|Z}}(L) := \limsup_{m\to\infty} \frac{\dim H^0(X|Z, \mathcal{O}(mL))}{m^d/d!},$$

which measures the asymptotic growth of these spaces.

On the other hand, as far as I know, the essential notion of the restricted version of the Carathéodory pseudo-volume form first appears in [11]. However it is not very easily comprehensible, and so we give the straightforward definition to compare with the restricted volume of the canonical bundle.

Henceforth, let X be an n-dimensional connected paracompact complex manifold and Z its d-dimensional irreducible subvariety which is possibly not closed unless further notice.

Definition 3.1. *When Z is smooth, the restricted Carathéodory pseudo-volume form $v^C_{X|Z}$ on Z is defined by setting*

$$v^C_{X|Z} := \sup\left\{ (f|_Z)^* \left(\frac{2^d}{d!(n+1)^d} (\sqrt{-1}\partial\overline{\partial}\log v_1)^d \right) \, ; \, f \in \mathrm{Hol}(X, \mathbb{B}^n) \right\}.$$

We remark several points about this definition. First, in the case $Z = X$, since $2^n/n!(n+1)^n \times (\sqrt{-1}\partial\overline{\partial}\log v_1)^n = v_1$, we have $v^C_{X|X} = v^C_X$. Second, we choose not $2^d/d!(d+1)^d$ but $2^d/d!(n+1)^d$ as a normalized constant in the definition (if we choose the other, we also have $v^C_{X|X} = v^C_X$). The reason is that if $X = \mathbb{B}^n$, and Y is a standard d-dimensional complex ball in $\mathbb{B}^n$, for instance $Y = \mathbb{B}^d = \{t \in \mathbb{B}^n \, ; \, t^{d+1} = \cdots = t^n = 0\}$, $v^C_{\mathbb{B}^n|\mathbb{B}^d}$ is the Poincaré volume form on the d-dimensional complex ball $\mathbb{B}^d$.

This definition is formulated to be able to obtain a restricted version of the comparison between the volume of the canonical bundle and the Carathéodory total volume, and it works well. But to calculate its curvature, we cannot simply use the formulation of the restricted version, and so we rewrite the formula (this is the one given in [11]):

Proposition 3.1. ([11]) *For any smooth Z, we have*

$$v^C_{X|Z} = \sup\left\{ (f|_Z)^* v_1^{(d)} \, ; \, f \in \mathrm{Hol}(X, \mathbb{B}^d) \right\},$$

where $v_1^{(d)}$ is the Poincaré volume form on $\mathbb{B}^d$.

This proposition suggests that we can define a global version of the restricted Carathéodory pseudo-volume form in the following fashion ([11]):

Definition 3.2. ([11]) *The restricted Carathéodory measure $\mu^C_{X|Z}$ on Z is defined as follows: for each Borel set $B \subset Z$, we put*

$$\mu^C_{X|Z}(B) := \sup\left\{ \sum_{i=1}^{\infty} \mu_1^{(d)}(f_i(B_i)) \, ; \, f_i \in \mathrm{Hol}(X, \mathbb{B}^d) \ (i \in \mathbb{N}) \text{ and } (B_i)_{i\in\mathbb{N}} : \text{mutually disjoint Borel sets of } Z \text{ s.t. } \bigcup_{i=1}^{\infty} B_i = B \right\},$$

where $\mu_1^{(d)}$ denotes the measure on $\mathbb{B}^d$ with $v_1^{(d)}$ as its density.

Furthermore, we introduce some of the basic properties of the restricted Carathéodory pseudo-volume form. All properties introduced here are ones which the original Carathéodory pseudo-volume form also has ([16]). First, by a similar reason to the original Carathéodory pseudo-volume form, we have

Proposition 3.2. ([11])

$$\mathbf{1}_{Z_{\mathrm{reg}}} v^C_{X|Z_{\mathrm{reg}}} = \mu^C_{X|Z}.$$

Since $\mathbb{B}^n$ is homogeneous, we can express the definition formula of $v^C_{X|Z}$ by the Ascoli-Arzelà theorem as follows: for any $x \in Z$, there exists $f_x \in \mathrm{Hol}(X, \mathbb{B}^n)$ with $f_x(x) = o$ such that

$$\begin{aligned}(v^C_{X|Z})_x &= \sup_{f(x)=o} \left\{ (f|_Z)^* \left(\frac{2^d}{d!(n+1)^d} (\sqrt{-1}\partial\overline{\partial} \log v_1)^d \right)_x \right\} \\ &= (f_x|_Z)^* \left(\frac{2^d}{d!(n+1)^d} (\sqrt{-1}\partial\overline{\partial} \log v_1)^d \right)_x .\end{aligned}$$

Therefore the pseudo-volume form is continuous on Z. Here we set a closed analytic subset $\mathrm{Z}(v^C_{X|Z}) := \{x \in Z\,;\, (v^C_{X|Z})_x = 0\}$ of Z.

A property of the restricted Carathéodory pseudo-volume form corresponding to the volume-decreasing property of the Carathéodory pseudo-volume form is the following:

Proposition 3.3. *Let Y be an n-dimensional complex manifold, and W its d-dimensional irreducible complex subvariety. Then for all $f \in \mathrm{Hol}(X, Y)$ such that $f(Z) \subset W$, we have*

$$f^* \mu^C_{Y|W} \leq \mu^C_{X|Z}.$$

In particular, $v^C_{X|Z}$ and $\mu^C_{X|Z}$ are invariant by holomorphic automorphisms of X preserving Z.

We begin to prove the restricted version of Corollary 1.1. We make its explicit statement.

Theorem 3.1. *Let X be an n-dimensional compact complex manifold and Z its d-dimensional irreducible closed subvariety. Take any Galois cover $\tilde{X} \xrightarrow{p} X$, and denote by $\tilde{Z}$ the pull-back of Z by p. Suppose that $\tilde{Z} \not\subset \mathrm{Z}(v^C_{\tilde{X}})$. Then we have*

$$\mathrm{vol}_{X|Z}(K_X) \geq \frac{d!(n+1)^d}{(4\pi)^d} \mu^C_{\tilde{X}|\tilde{Z}}(Z).$$

In fact, we note that X is Moishezon because of the assumption $\mathrm{Z}(v^C_{\tilde{X}}) \neq \emptyset$, and so a simple observation leads that the non-projective case reduces to the projective case by Moishezon's theorem and the Riemann extension theorem. A key of the proof of Theorem 3.1 is the following inequalities:

Theorem 3.2. *Let X be an n-dimensional complex manifold and Z its d-dimensional closed submanifold not contained in $\mathrm{Z}(v^C_X)$. Then we have the following inequality on Z:*

$$\left(\Theta_{(v^C_X)^{-1}}|_Z\right)^d_{\mathrm{ac}} = \left(\sqrt{-1}\partial\overline{\partial}\log v^C_X|_Z\right)^d_{\mathrm{ac}} \geq \frac{d!(n+1)^d}{2^d} v^C_{X|Z}. \tag{2}$$

Moreover suppose that X has a smooth compact projective quotient and Z is invariant under the deck transformation. Then we have

$$\left\langle \left(\Theta_{(v^C_X)^{-1}}|_Z\right)^d \right\rangle = \left\langle \left(\sqrt{-1}\partial\overline{\partial}\log v^C_X|_Z\right)^d \right\rangle \geq \frac{d!(n+1)^d}{2^d} v^C_{X|Z}, \tag{3}$$

where $\langle\cdot\rangle$ means the non-pluripolar Monge-Ampère product.

Proof of Theorem 3.2. The proof of (2) is almost the same as the one of Theroem 1.3 and so we omit the detail here.

Next we prove (3). First note that we certainly suppose that the compact quotient is projective, and more that v^C_X is an invariant and continuous pseudo-volume form with an invariant analytic subset $\mathrm{Z}(v^C_X)$. So it is known ([3]) that the non-pluripolar product of the restriction $\Theta_{(v^C_X)^{-1}}|_Z = \sqrt{-1}\partial\overline{\partial}\log\left(v^C_X|_Z\right)$ to Z of the curvature current $\Theta_{(v^C_X)^{-1}}$ is well-defined and is merely the zero extension to Z of the product $\left(\mathbf{1}_{Z\setminus \mathrm{Z}(v^C_X)}\sqrt{-1}\partial\overline{\partial}\log\left(v^C_X|_Z\right)\right)^d$ on $Z \setminus \mathrm{Z}(v^C_X)$ due to Bedford-Taylor. Therefore we can easily observe that over Z

$$\left\langle \left(\sqrt{-1}\partial\overline{\partial}\log\left(v^C_X|_Z\right)\right)^d \right\rangle \geq \left(\sqrt{-1}\partial\overline{\partial}\log\left(v^C_X|_Z\right)\right)^d_{\mathrm{ac}},$$

which by (2) implies the inequality (3) which we want. Indeed, we have to prove this inequality only locally on $Z \setminus \mathrm{Z}(v^C_X)$ in which

$$\left\langle \left(\sqrt{-1}\partial\overline{\partial}\log\left(v^C_X|_Z\right)\right)^d \right\rangle = \left(\sqrt{-1}\partial\overline{\partial}\log\left(v^C_X|_Z\right)\right)^d$$

by the above remark. For a standard regularization kernel $(\rho_\epsilon)_{\epsilon>0}$, we have locally

$$\left(\sqrt{-1}\partial\overline{\partial}\log\left(v^C_X|_Z\right) * \rho_\epsilon\right)^d \xrightarrow{\epsilon\to 0} \left(\sqrt{-1}\partial\overline{\partial}\log\left(v^C_X|_Z\right)\right)^d,$$

and we also have locally

$$\left(\sqrt{-1}\partial\overline{\partial}\log\left(v_X^C|_Z\right)*\rho_\epsilon\right)^d \xrightarrow{\epsilon\to 0} \left(\sqrt{-1}\partial\overline{\partial}\log\left(v_X^C|_Z\right)\right)_{\mathrm{ac}}^d \quad \text{a.e.}$$

by Lebesgue's theorem. Therefore by Fatou's lemma, we obtain

$$\left(\sqrt{-1}\partial\overline{\partial}\log\left(v_X^C|_Z\right)\right)^d \geq \left(\sqrt{-1}\partial\overline{\partial}\log\left(v_X^C|_Z\right)\right)_{\mathrm{ac}}^d.$$

We complete the proof of Theorem 3.2. □

As for Theorem 3.1, this is directly obtained by another expression formula of the restricted volume with respect to the non-pluripolar product due to Hisamoto ([12]) and Matsumura ([18]) besides Theorem 3.2.

Theorem 3.3. *Let X be an n-dimensional projective manifold, L a big line bundle over X and Z a d-dimensional closed irreducible subvariety of X. Furthermore suppose that $Z \not\subset \mathbb{B}_+(L)$. Then we have*

$$\mathrm{vol}_{X|Z}(L) = \sup_T \int_{Z_{\mathrm{reg}}} \langle (T|_{Z_{\mathrm{reg}}})^d \rangle,$$

where T runs though all positive $T \in c_1(L)$ which have small unbounded loci and whose unbounded loci do not contain Z.

Here an analytic subset $\mathbb{B}_+(L)$ of X is defined as

$$\mathbb{B}_+(L) = \left\{ x \in X; \begin{array}{c} \text{there is no singular Hermitian metric } h \text{ on } L \\ \text{such that} \quad h: \text{smooth around } x \\ \qquad\qquad \Theta_h: \text{strictly positive on } X \end{array} \right\},$$

which is called the augmented base locus of L. Furthermore, the unbounded locus of a positive $T \in c_1(L)$ is the unbounded locus of its potential, and such T has a small unbounded locus if and only if its unbounded locus is contained in certain analytic subset of X.

Proof of Theorem 3.1. In the situation of Theorem 3.1, the unbounded locus of the curvature current $\sqrt{-1}\partial\overline{\partial}\log v_{\tilde{X}}^C$ is $\mathrm{Z}(v_{\tilde{X}}^C)$. Moreover, Z is not contained in the augmented base locus $\mathbb{B}_+(K_X)$ of the canonical bundle K_X since we have

Lemma 3.1.

$$\mathbb{B}_+(K_X) \subset \mathrm{Z}(v_{\tilde{X}}^C).$$

This is proved by the standard L^2 estimates for the $\overline{\partial}$ operator as follows ([7]). It is sufficient to show that for any point $x \notin \mathrm{Zero}(v^C_{\tilde{X}})$ and a smooth sufficiently ample divisor H not containing x, there exists a global section of $(m+1)K_X - H$ nonvanishing at x for a sufficiently large $m \in \mathbf{N}$. Indeed, we consider the singular hermitian metric $(v^C_{\tilde{X}})^{-m}$ on $K_X^{\otimes m}$ over a Stein manifold $X \setminus H$ and note that its curvature current satisfies the positivity property stated in the latter of Theorem 2.2. Then by the L^2 estimate, we find a holomorphic section of $(m+1)K_X$ nonvanishing at x and vanishing along H, which is required.

Therefore $\frac{1}{2\pi}\Theta_{(v^C_{\tilde{X}})^{-1}}$ and Z satisfy the hypothesis of T and Z in Theorem 3.3, respectively. Finally we conclude by (3) of Theorem 3.2 that

$$\mathrm{vol}_{X|Z}(K_X) \geq \int_{Z_{\mathrm{reg}}} \left\langle \left(\frac{1}{2\pi} \Theta_{(v^C_{\tilde{X}})^{-1}}|_{Z_{\mathrm{reg}}} \right)^d \right\rangle \geq \frac{d!(n+1)^d}{(4\pi)^d} \mu^C_{\tilde{X}|\tilde{Z}}(Z).$$

□

We remark that we can construct a counterexample to the volume comparison when the condition $\tilde{Z} \not\subset \mathrm{Zero}(v^C_{\tilde{X}})$ is removed. Let C be a smooth projective curve whose genus is greater than 1, $p \in C$ and $\varphi : X \to C^3$ the blow-up of the smooth 3-fold $C^3 = C \times C \times C$ along $C = C \times p \times p$. We take the closed curve $Z \subset X$ in the exceptional divisor $E = \varphi^{-1}(C) = C \times \mathbb{P}^1$ of φ which is defined as $Z = C \times (1:0) \subset C \times \mathbb{P}^1$, and put $\tilde{Z} = \pi^{-1}(Z) \subset \tilde{X}$ which is regarded as a countably disjoint union of the unit disk Δ. Then we can have $v^C_{\tilde{X}|\tilde{Z}} = v_1$ on any connected component Δ of $\tilde{Z}$, and $\mathrm{vol}_{X|Z}(K_X) = 0$. As a consequence, the volume comparison like Theorem 3.1 surely fails for this X and Z.

Finally, we state the proposition about the curvature of the restricted Carathéodory pseudo-volume from, which is a generalization of Theorem 1.3. The proof is completely the same as Theorem 1.3 by using Proposition 3.1, the reformulation of the restricted Carathéodory pseudo-volume form.

Proposition 3.4. *For smooth Z, we have*

$$K_{v^C_{X|Z}} \leq -1$$

and

$$\sqrt{-1}\partial\overline{\partial}\log v^C_{X|Z} > 0 \quad \text{on } Z \setminus \mathrm{Z}(v^C_{X|Z}).$$

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