Cooperation principle, stability and bifurcation in random complex dynamics

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Abstract

We investigate the random dynamics of rational maps and the dynamics of semigroups of rational maps on the Riemann sphere $\hat{\mathbb{C}}$. We show that regarding random complex dynamics of polynomials, generically, the chaos of the averaged system disappears, due to the automatic cooperation of the generators. We investigate the iteration and spectral properties of transition operators acting on the space of (Hölder) continuous functions on $\hat{\mathbb{C}}$. We also investigate the stability and bifurcation of random complex dynamics. We show that the set of stable systems is open and dense in the space of random dynamics of polynomials. Moreover, we prove that for a stable system, there exist only finitely many minimal sets, each minimal set is attracting, and the orbit of a Hölder continuous function on $\hat{\mathbb{C}}$ under the transition operator tends exponentially fast to the finite-dimensional space U of finite linear combinations of unitary eigenvectors of the transition operator. Thus the spectrum of the transition operator acting on the space of Hölder continuous functions has a gap between the set of unitary eigenvalues and the rest. Combining this with the perturbation theory for linear operators, we obtain that for a stable system constructed by a finite family of rational maps, the projection to the space U depends real-analytically on the probability parameters. By taking a partial derivative of the function of probability of tending to a minimal set with respect to a probability parameter, we obtain a complex analogue of the Takagi function. Many new phenomena which can hold in random complex dynamics but cannot hold in the iteration of a single rational map are found and systematically investigated.

1 Introduction

The details of this talk are included in the author's papers [48, 50].

One motivation for research in complex dynamical systems (for the introductory texts, see [23, 3]) is to describe some mathematical models on ethology. For example, the behavior of the population of a certain species can be described by the dynamical system associated with iteration of a polynomial $f(z) = az(1-z)$ (cf. [9]). However, when there is a change in the natural environment, some species have several strategies to survive in nature. From this point of view, it is very natural and important not only to consider the dynamics of iteration, where the same survival strategy (i.e., function) is repeatedly applied, but also to consider random dynamics, where a new strategy might be applied at each time step. Another motivation for research in complex

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dynamics is Newton's method to find a root of a complex polynomial, which often is expressed as the dynamics of a rational map g on $\hat{\mathbb{C}}$ with $\deg(g) \geq 2$, where $\deg(g)$ denotes the degree of g. We sometimes use computers to analyze such dynamics, and since we have some errors at each step of the calculation in the computers, it is quite natural to investigate the random dynamics of rational maps. In various fields, we have many mathematical models which are described by the dynamical systems associated with polynomial or rational maps. For each model, it is natural and important to consider a randomized model, since we always have some kind of noise or random terms in nature. The first study of random complex dynamics was given by J. E. Fornaess and N. Sibony ([10]). They mainly investigated random dynamics generated by small perturbations of a single rational map. For research on random complex dynamics of quadratic polynomials, see [4, 5, 6, 7, 8, 11]. For research on random dynamics of polynomials (of general degrees), see the author's works [45, 46, 47, 49, 48, 50]. For the texts of general theory of random dynamical systems (on real manifolds), see [1, 20], though these texts do not deal with random complex dynamics. There have been no textbooks of random complex dynamics yet.

In order to investigate random complex dynamics, it is very natural to study the dynamics of associated rational semigroups. In fact, it is a very powerful tool to investigate random complex dynamics, since random complex dynamics and the dynamics of rational semigroups are related to each other very deeply. The first study of dynamics of rational semigroups was conducted by A. Hinkkanen and G. J. Martin ([14]), who were interested in the role of the dynamics of polynomial semigroups (i.e., semigroups of non-constant polynomial maps) while studying various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group ([12]), who studied such semigroups from the perspective of random dynamical systems. Since the Julia set $J(G)$ of a finitely generated rational semigroup $G = \langle h_1, \ldots, h_m \rangle$ has "backward self-similarity," i.e., $J(G) = \bigcup_{j=1}^{m} h_j^{-1}(J(G))$ (see [34, Lemma 1.1.4]), the study of the dynamics of rational semigroups can be regarded as the study of "backward iterated function systems," and also as a generalization of the study of self-similar sets in fractal geometry. For recent work on the dynamics of rational semigroups, see the author's papers [34]–[50], and [27, 28, 29, 30, 31, 32, 33, 51, 52, 53, 54]. ([31] is a very nice introductory paper which gives a short and elementary proof of the density of repelling fixed points in the Julia set of a rational semigroup.)

In this talk, by combining several results from [48] and many new ideas, we investigate the random complex dynamics and the dynamics of rational semigroups. In the usual iteration dynamics of a single rational map g with $\deg(g) \geq 2$, we always have a non-empty chaotic part, i.e., in the Julia set $J(g)$ of g, which is a perfect set, we have sensitive initial values and dense orbits. Moreover, for any ball B with $B \cap J(g) \neq \emptyset$, $g^n(B)$ expands as $n \to \infty$ (g^n denotes the n-th iterate of g). Regarding random complex dynamics, it is natural to ask the following question. Do we have a kind of "chaos" in the averaged system? Or do we have no chaos? How do many kinds of maps in the system interact? What can we say about stability and bifurcation? Since the chaotic phenomena hold even for a single rational map, one may expect that in random dynamics of rational maps, most systems would exhibit a great amount of chaos. However, it turns out that this is not true. One of the main purposes of this talk is to present that for a generic system of random complex dynamics of polynomials, many kinds of maps in the system "automatically" cooperate so that they make the chaos of the averaged system disappear, even though the dynamics of each map in the system have a chaotic part. We call this phenomenon the "**cooperation principle**". Moreover, we prove that for a generic system, we have a kind of stability (see Theorem C). We remark that the chaos disappears in the C^0 "sense", but under certain conditions, the chaos remains in the C^β "sense", where C^β denotes the space of β-Hölder continuous functions with exponent $\beta \in (0, 1)$ (see Appendix).

Before going into the details of random complex dynamics, We consider the random dynamical systems on $\mathbb{R}$. In order to do that, let us recall the definition of the devil's staircase (the Cantor function), Lebesgue' singular functions, and the Takagi function. Generally, the devil's staircase, Lebesgue's singular functions, and the Takagi function are defined as bounded functions on $[0, 1]$ which satisfy some kinds of functional equations and boundary conditions ([55, 2]). More precisely (the following definitions look a little bit different from those in [55, 2], but it turns out that they are equivalent to those in [55, 2]), the devil's staircase is defined as the restriction $\varphi|_{[0,1]}$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is the unique bounded function which satisfies

$$\frac{1}{2}\varphi(3x) + \frac{1}{2}\varphi(3x-2) = \varphi(x) \ (\forall x \in \mathbb{R}), \ \varphi|_{(-\infty,0]} \equiv 0, \ \varphi|_{[1,+\infty)} \equiv 1, \tag{1}$$

and for each $0 < a < 1$ with $a \neq 1/2$, Lebesgue's singular function L_a is defined as the restriction $\psi_a|_{[0,1]}$, where $\psi_a : \mathbb{R} \to \mathbb{R}$ is the unique bounded function which satisfies

$$a\psi_a(2x) + (1-a)\psi_a(2x-1) = \psi_a(x) \ (\forall x \in \mathbb{R}), \ \psi_a|_{(-\infty,0]} \equiv 0, \ \psi_a|_{[1,+\infty)} \equiv 1, \tag{2}$$

and the Takagi function is defined as the restriction $\phi/2|_{[0,1]}$, where $\phi : \mathbb{R} \to \mathbb{R}$ is the unique bounded function which satisfies

$$\frac{1}{2}\phi(2x) + \frac{1}{2}\phi(2x-1) + \psi_{1/2}(2x) - \psi_{1/2}(2x-1) = \phi(x) \ (\forall x \in \mathbb{R}), \ \phi|_{(-\infty,0]\cup[1,+\infty)} \equiv 0. \tag{3}$$

We now give a (relatively new, uncommon) interpretation for these functions in terms of random dynamical systems on $\mathbb{R}$. Let $f_1(x) := 3x, \ f_2(x) := 3(x-1)+1 \ (x \in \mathbb{R})$. We consider the random dynamics on $\mathbb{R}$ such that at every step we choose f_1 with probability $1/2$ and f_2 with probability $1/2$. Let $\tilde{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be the two-point compactification of $\mathbb{R}$. For each $x \in \mathbb{R}$, let $T_{+\infty}(x)$ be the probability of tending to $+\infty$ starting with the initial value $x \in \mathbb{R}$. As the author pointed out in [49, 48], we see that the restriction $T_{+\infty}|_{[0,1]} : [0,1] \to [0,1]$ is equal to the devil's staircase (or the Cantor function) (Figure 1). The devil's staircase satisfies the following properties:

(a) It is continuous on $[0,1]$.

(b) It varies precisely on the Cantor middle third set C (a kind of thin fractal set), i.e., $T'_{+\infty}(x) = 0$ for $x \in \mathbb{R} \setminus C$ and $T_{+\infty}|_U$ is not constant for each open subset U of $\mathbb{R}$ with $\mathbb{R} \cap C \neq \emptyset$.

(c) It is monotone.

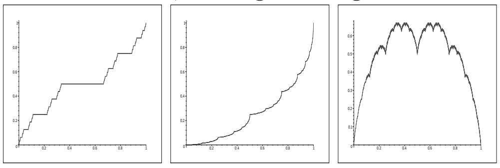
Similarly, let $g_1(x) := 2x, \ g_2(x) := 2(x-1)+1 \ (x \in \mathbb{R})$. For each $0 < a < 1$, we consider the random dynamics on $\mathbb{R}$ such that at every step we choose g_1 with probability a and g_2 with probability $1-a$. Let $T_{+\infty,a}(x)$ be the probability of tending to $+\infty$ starting with the initial value $x \in \mathbb{R}$. As the author pointed out in [49, 48], we see that the function $T_{+\infty,a}|_{[0,1]} : [0,1] \to [0,1]$ is equal to Lebesgue's singular function $L_a : [0,1] \to [0,1]$ with parameter a (Figure 1). The function L_a is continuous and monotone on $[0,1]$, and if $a \neq 1/2$, L_a has the following singular property: for almost every point $x \in [0,1]$ with respect to the one-dimensional Lebesgue measure, L_a is differentiable at x and the derivative is zero. Moreover, Sekiguchi and Shiota showed that for each fixed $x \in [0,1]$, the function $a \mapsto L_a(x)$ is real-analytic in $(0,1)$ ([26]), and Hata and Yamaguti showed that the function $x \mapsto (1/2) \cdot (\partial L_a(x)/\partial a)|_{a=1/2}$ on $[0,1]$ is equal to the Takagi function ([13], Figure 1).

Thus, the devil's staircase and Lebesgue's singular functions can be defined in terms of random dynamics on $\mathbb{R}$, that is, they can be defined as the functions of probability of tending to $+\infty$. Moreover, the Takagi function can be defined as the partial derivative with respect to the probability parameter.

We remark that in most of the literature, the theory of random dynamical systems has not been used directly to investigate these singular functions on the interval, although some researchers have used it implicitly. However, as the author points out, it is very natural and straightforward to use the theory of random dynamical systems in the study of these singular functions.

In this talk, we consider a complex analogue of the above story. Moreover, we consider the disappearance of chaos, stability and bifurcation in random complex dynamics.

Figure 1: (From left to right) the devil's staircase, Lebesgue's singular function, the Takagi function



2 Preliminaries

Definition 2.1.

- We denote by $\hat{\mathbb{C}}(:= \mathbb{C} \cup \{\infty\} \cong \mathbb{CP}^1 \cong S^2)$ the Riemann sphere and denote by d the spherical distance on $\hat{\mathbb{C}}$.
- We set Rat:=$\{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h$ is a non-const. rational map$\}$ endowed with the distance η defined by $\eta(f,g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$.We set $\text{Rat}_+ := \{g \in \text{Rat} \mid \deg(g) \geq 2\}$.
- We set $\mathcal{P} := \{g : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid g \text{ is a polynomial map}, \deg(g) \geq 2\}$ endowed with the relative topology from Rat.
- Note that Rat and $\mathcal{P}$ are semigroups where the semigroup operation is functional composition.
- A subsemigroup G of Rat is called a **rational semigroup**.
- A subsemigroup G of $\mathcal{P}$ is called a **polynomial semigroup**.

Definition 2.2. Let G be a rational semigroup.

- We set $F(G) := \{z \in \hat{\mathbb{C}} \mid \exists \text{ nbd } U \text{ of } z \text{ s.t. } G \text{ is equicontinuous on } U\}$.
This is called the **Fatou set** of G. (For the definition of equicontinuity, see [3].)
- We set $J(G) := \hat{\mathbb{C}} \setminus F(G)$. This is called the **Julia set** of G.
- If G is generated by a subset Λ of Rat, then we write $G = \langle \Lambda \rangle$.

Definition 2.3. For a topological space X, we denote by $\mathfrak{M}_1(X)$ the space of all Borel probability measures on X endowed with the topology such that
"$\mu_n \to \mu$" $\Leftrightarrow$ "for each bounded continuous function $\varphi : X \to \mathbb{R}$, $\int_X \varphi \, d\mu_n \to \int_X \varphi \, d\mu$."

Remark 2.4. If X is a compact metric space, then $\mathfrak{M}_1(X)$ is a compact metrizable space.

From now on, we take a $\tau \in \mathfrak{M}_1(\text{Rat})$ and we consider the (i.i.d.) random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a map $h \in \text{Rat}$ according to τ. This determines a time-discrete Markov process with time-homogeneous transition probabilities on the phase space $\hat{\mathbb{C}}$ such that for each $x \in \hat{\mathbb{C}}$ and for each Borel measurable subset A of $\hat{\mathbb{C}}$, the transition probability $p(x, A)$ of the Markov process from x to A is defined as $p(x, A) = \tau(\{g \in \text{Rat} \mid g(x) \in A\})$.

Definition 2.5. Let $\tau \in \mathfrak{M}_1(\text{Rat})$.

(1) We set $C(\hat{\mathbb{C}}) := \{\varphi : \hat{\mathbb{C}} \to \mathbb{C} \mid \varphi \text{ is conti.}\}$ endowed with the sup. norm $\|\cdot\|_\infty$.

(2) Let $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ be the operator defined by

$$M_\tau(\varphi)(z) := \int_{\text{Rat}} \varphi(g(z)) \, d\tau(g), \ \forall \varphi \in C(\hat{\mathbb{C}}), \forall z \in \hat{\mathbb{C}}.$$

(3) Let $M_\tau^* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ be the dual of M_τ. That is, for each $\rho \in \mathfrak{M}_1(\hat{\mathbb{C}})$ and for each $\varphi \in C(\hat{\mathbb{C}})$,

$$\int \varphi \ d(M_\tau^*(\rho)) := \int M_\tau(\varphi) \ d\rho.$$

(Remark: M_τ^* can be regarded as the "averaged map" of $\operatorname{supp}\tau$, where $\operatorname{supp}\tau$ denotes the topological support of τ. More precisely, let $\Phi : \hat{\mathbb{C}} \to \mathfrak{M}_1(\hat{\mathbb{C}})$ be the topological embedding defined by: $\Phi(z) := \delta_z$, where δ_z denotes the Dirac measure at z. Using this topological embedding $\Phi : \hat{\mathbb{C}} \to \mathfrak{M}_1(\hat{\mathbb{C}})$, we regard $\hat{\mathbb{C}}$ as a compact subset of $\mathfrak{M}_1(\hat{\mathbb{C}})$. If $h \in \mathrm{Rat}$, then we have $M_{\delta_h}^* \circ \Phi = \Phi \circ h$ on $\hat{\mathbb{C}}$ (i.e., $M_{\delta_h}^*(\delta_z) = \delta_{h(z)}$ for each $z \in \hat{\mathbb{C}}$). Moreover, for a general $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, for each $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$, we have $M_\tau^*(\mu) = \int M_{\delta_h}^*(\mu) d\tau(h) = \int h_*(\mu) d\tau(h)$, where $h_*(\mu)$ denotes the Borel probability measure on $\hat{\mathbb{C}}$ such that $h_*(A) := \mu(h^{-1}(A))$ for each Borel subset A of $\hat{\mathbb{C}}$. Therefore, for a general $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, the map $M_\tau^* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ can be regarded as the "averaged map" on the extension $\mathfrak{M}_1(\hat{\mathbb{C}})$ of $\hat{\mathbb{C}}$.)

(4) We set
$F_{meas}(\tau) := \{\mu \in \mathfrak{M}_1(\hat{\mathbb{C}}) \mid \exists$ nbd B of μ in $\mathfrak{M}_1(\hat{\mathbb{C}})$ s.t.
$\{(M_\tau^*)^n|_B : B \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n\in\mathbb{N}}$ is equiconti. on $B\}$.

(5) Let $\mathcal{U}_\tau$ be the space of all finite linear combinations of unitary eigenvectors of $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$, where an eigenvector is said to be unitary if the absolute value of the corresponding eigenvalue is 1.

(6) Let $\mathcal{B}_{0,\tau} := \{\varphi \in C(\hat{\mathbb{C}}) \mid M_\tau^n(\varphi) \to 0 \text{ as } n \to \infty\}$.

(7) Let $\tilde{\tau} := \otimes_{j=1}^{\infty} \tau \in \mathfrak{M}_1((\mathrm{Rat})^{\mathbb{N}})$.

(8) Let G_τ be the rational semigroup generated by $\operatorname{supp}\tau$.

(9) Let G be a rational semigroup. We say that a non-empty compact subset K of $\hat{\mathbb{C}}$ is a minimal set of G in $\hat{\mathbb{C}}$ if K is minimal in $\{L \subset \hat{\mathbb{C}} \mid \emptyset \neq L \text{ is compact}, \forall g \in G, g(L) \subset L\}$ w.r.t. $\subset$. Moreover, we set $\mathrm{Min}(G, \hat{\mathbb{C}}) := \{L \mid L \text{ is a minimal set of } G \text{ in } \hat{\mathbb{C}}\}$.

(10) For a minimal set L of G_τ in $\hat{\mathbb{C}}$ and a point $z \in \hat{\mathbb{C}}$, we set $T_{L,\tau}(z) := \tilde{\tau}(\{\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{Rat})^{\mathbb{N}} \mid d(\gamma_n \cdots \gamma_1(z), L) \to 0 \text{ as } n \to \infty\})$. This is **the probability of tending to** L **starting with the initial value** $z \in \hat{\mathbb{C}}$.

The following is the key to investigating the random complex dynamics.

Definition 2.6. Let G be a rational semigroup. We set

$$J_{\ker}(G) := \bigcap_{h\in G} h^{-1}(J(G)).$$

This is called the **kernel Julia set** of G.

By the forward invariance of $J_{\ker}(G)$ under each map of G, Montel's theorem, and the fact that $\infty \in F(\langle\Gamma\rangle)$ for a compact subset Γ of $\mathcal{P}$, we obtain the following.

Lemma 2.7. *Let $\tau \in \mathfrak{M}_1(\mathcal{P})$ be such that $\operatorname{supp}\tau$ is compact. Suppose that for each $z \in \mathbb{C}$, there exists a holomorphic family $\{g_\lambda\}_{\lambda\in\Lambda}$ of polynomial maps such that $\bigcup_{\lambda\in\Lambda}\{g_\lambda\} \subset \operatorname{supp}\tau$ and such that $\lambda \mapsto g_\lambda(z)$ is not constant on Λ. Then, $J_{\ker}(G_\tau) = \emptyset$. Here, a family $\{g_\lambda\}_{\lambda\in\Lambda}$ of rational (resp. polynomial) maps is said to be a holomorphic family of rational (resp. polynomial) maps if Λ is a finite-dimensional complex manifold and $(z, \lambda) \in \hat{\mathbb{C}} \times \Lambda \mapsto g_\lambda(z) \in \hat{\mathbb{C}}$ is holomorphic on $\hat{\mathbb{C}} \times \Lambda$.*

For example, let $\tau \in \mathfrak{M}_1(\mathcal{P})$ be such that $\operatorname{supp} \tau$ is compact. If there exists an $f_0 \in \mathcal{P}$ and a non-empty open subset U of $\mathbb{C}$ such that $\{f_0 + c \mid c \in U\} \subset \operatorname{supp} \tau$, then $J_{\ker}(G_\tau) = \emptyset$.

Thus, we can say that mostly $J_{\ker}(G_\tau) = \emptyset$. (Note: if G is a group or commutative group, then $J_{\ker}(G) = J(G)$. Thus if G is a non-elementary Kleinian group or G is generated by a single rational map g with $g \geq 2$, then $J_{\ker}(G) = J(G) \neq \emptyset$.)

3 Results

Theorem 3.1 (Theorem A, Cooperation Principle and Disappearance of Chaos). *Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ be such that $\operatorname{supp} \tau$ is compact. Suppose $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$.*
(note: if $\exists g \in \operatorname{supp}\tau$ with $\deg(g) \geq 2$, then $J(G_\tau) \neq \emptyset$.)
Then, we have all of the following (1)–(9).

(1) *$F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$. For $\tilde{\tau}$-a.e. $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{Rat})^{\mathbb{N}}$, the 2-dim. Leb. meas. of $J_\gamma := \{z \in \hat{\mathbb{C}} \mid \forall \text{nbd } U \text{ of } z, \{\gamma_n \cdots \gamma_1|_U : U \to \hat{\mathbb{C}}\}_{n \in \mathbb{N}} \text{ is not equiconti. on } U\}$. is zero.*

(2) *$\mathcal{B}_{0,\tau}$ is a closed subspace of $C(\hat{\mathbb{C}})$ and $C(\hat{\mathbb{C}}) = \mathcal{U}_\tau \oplus \mathcal{B}_{0,\tau}$.*

(3) $1 \leq \dim_{\mathbb{C}} \mathcal{U}_\tau < \infty$.

(4) *For each $\varphi \in \mathcal{U}_\tau$ and for each connected component U of $F(G_\tau)$, $\varphi|_U$ is constant.*

(5) *$\exists \alpha \in (0,1)$ s.t. $\forall \varphi \in \mathcal{U}_\tau$, φ is α-Hölder continuous on $\hat{\mathbb{C}}$.*

(6) *For $\forall z \in \hat{\mathbb{C}}$, $\exists \mathcal{A}_z \subset (\mathrm{Rat})^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{A}_z) = 1$ with the following property.*

  - *$\forall \gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{A}_z$, $\exists \delta = \delta(z, \gamma) > 0$ s.t. $\operatorname{diam} \gamma_n \cdots \gamma_1(B(z, \delta)) \to 0$ as $n \to \infty$, where* diam *denotes the diameter w.r.t. the spherical distance.*

(7) *There exist at least one and at most finitely many minimal sets of G_τ in $\hat{\mathbb{C}}$.*

(8) *Let S_τ be the union of minimal sets of G_τ in $\hat{\mathbb{C}}$. Then $\forall z \in \hat{\mathbb{C}}$ $\exists \mathcal{C}_z \subset (\mathrm{Rat})^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{C}_z) = 1$ s.t. $\forall \gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{C}_z$, $d(\gamma_n \cdots \gamma_1(z), S_\tau) \to 0$ as $n \to \infty$.*

(9) *Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$. Then $M_\tau(T_{L,\tau}) = T_{L,\tau}$ and $T_{L,\tau} \in \mathcal{U}_\tau$. Moreover, for each $z \in \hat{\mathbb{C}}$, $\sum_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} T_{L,\tau}(z) = 1$.*

Remark 3.2. Theorem A describes new phenomena which cannot hold in the usual iteration dynamics of a single $g \in \mathrm{Rat}$ with $\deg(g) \geq 2$. For example, $F_{meas}(\delta_g) \neq \mathfrak{M}_1(\hat{\mathbb{C}})$.

We remark that in 1983, by numerical experiments, K. Matsumoto and I. Tsuda ([21]) observed that if we add some uniform noise to the dynamical system associated with iteration of a chaotic map on the unit interval $[0, 1]$, then under certain conditions, the quantities which represent chaos (e.g., entropy, Lyapunov exponent, etc.) decrease. More precisely, they observed that the entropy decreases and the Lyapunov exponent turns negative. They called this phenomenon "noise-induced order", and many physicists have investigated it by numerical experiments, although there has been only a few mathematical supports for it.

We now consider a sufficient condition for τ to be $J_{\ker}(G_\tau) = \emptyset$.

Definition 3.3. Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ be such that $\operatorname{supp} \tau$ is compact. We say that τ is **mean stable** if there exist non-empty open subsets U, V of $F(G_\tau)$ and a number $n \in \mathbb{N}$ such that all of the following hold.

(1) $\overline{V} \subset U \subset \overline{U} \subset F(G_\tau)$.

(2) $\forall \gamma = (\gamma_1, \gamma_2, \ldots) \in (\operatorname{supp} \tau)^{\mathbb{N}}$, $(\gamma_n \circ \cdots \circ \gamma_1)(\overline{U}) \subset V$.

(3) $\forall z \in \hat{\mathbb{C}}$, $\exists g \in G_\tau$ s.t. $g(z) \in U$.

Remark 3.4. If τ is mean stable, then $J_{\ker}(G_\tau) = \emptyset$.

We can see that τ is mean stable if and only if the cardinality of the set of all minimal sets of G_τ in $\hat{\mathbb{C}}$ is finite and each minimal set L is "attracting", i.e., there exists an open subset W_L of $F(G_\tau)$ with $L \subset W_L$ and an $\epsilon > 0$ such that for each $z \in W_L$ and for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\operatorname{supp}\tau)^{\mathbb{N}}$, $d(\gamma_n \cdots \gamma_1(z), L) \to 0$ and $\operatorname{diam}(\gamma_n \cdots \gamma_1(B(z,\epsilon))) \to 0$ as $n \to \infty$. Thus, the notion "mean stability" of random complex dynamics can be regarded as a kind of analogy of "hyperbolicity" of the usual iteration dynamics of a single rational map.

Definition 3.5. Let $\mathcal{Y}$ be a closed subset of Rat. Let

$$\mathfrak{M}_{1,c}(\mathcal{Y}) := \{\tau \in \mathfrak{M}_1(\mathcal{Y}) \mid \operatorname{supp}\tau \text{ is compact}\}.$$

Let $\mathcal{O}$ be the topology in $\mathfrak{M}_{1,c}(\mathcal{Y})$ such that $\tau_n \to \tau$ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ if and only if

- $\int \varphi\, d\tau_n \to \int \varphi\, d\tau$ for each bounded continuous function $\varphi : \mathcal{Y} \to \mathbb{R}$, and
- $\operatorname{supp}\tau_n \to \operatorname{supp}\tau$ with respect to the Hausdorff metric in the space of all non-empty compact subsets of $\mathcal{Y}$.

Definition 3.6. let $\mathcal{Y}$ be a subset of Rat. We say that $\mathcal{Y}$ satisfies condition $(*)$ if $\mathcal{Y}$ is closed in Rat and at least one of the following (1) and (2) holds:

(1) for each $(z_0, h_0) \in \hat{\mathbb{C}} \times \mathcal{Y}$, there exists a holomorphic family $\{g_\lambda\}_{\lambda\in\Lambda}$ of rational maps with $\bigcup_{\lambda\in\Lambda}\{g_\lambda\} \subset \mathcal{Y}$ and an element $\lambda_0 \in \Lambda$, such that, $g_{\lambda_0} = h_0$ and $\lambda \mapsto g_\lambda(z_0)$ is non-constant in any neighborhood of λ_0.

(2) $\mathcal{Y} \subset \mathcal{P}$ and for each $(z_0, h_0) \in \mathbb{C}\times\mathcal{Y}$, there exists a holomorphic family $\{g_\lambda\}_{\lambda\in\Lambda}$ of polynomial maps with $\bigcup_{\lambda\in\Lambda}\{g_\lambda\} \subset \mathcal{Y}$ and an element $\lambda_0 \in \Lambda$ such that $g_{\lambda_0} = h_0$ and $\lambda \mapsto g_\lambda(z_0)$ is non-constant in any neighborhood of λ_0.

Example 3.7. Rat, Rat_+, $\mathcal{P}$, and $\{z^d + c \mid c \in \mathbb{C}\}$ $(d \in \mathbb{N}, d \geq 2)$ satisfy condition $(*)$.

Theorem 3.8 (Theorem B, Density of Mean Stable Systems). *Let $\mathcal{Y}$ be a subset of $\mathcal{P}$ satisfying $(*)$. Then we have the following.*

(1) $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau$ *is mean stable*$\}$ *is open and dense in* $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$.

(2) $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau$ *is mean stable and* $\sharp\operatorname{supp}\tau < \infty\}$ *is dense in* $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$.

We remark that in the study of iteration of a single rational map, we have a very famous conjecture (HD conjecture, see [22, Conjecture 1.1]) which states that hyperbolic rational maps are dense in the space of rational maps. Theorem B solves this kind of problem in the study of random dynamics of complex polynomials (see the comments after Remark 3.4).

Theorem 3.9. *Let $\mathcal{Y}$ be a subset of* Rat_+ *satisfying condition* $(*)$*. Then, the set*

$$\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau \textit{ is mean stable } \} \cup \{\rho \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \operatorname{Min}(G_\rho, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}, J(G_\rho) = \hat{\mathbb{C}}\}$$

is dense in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$.

For the proofs of Theorems B and 3.9, we need to investigate and classify the minimal sets of $\langle\Gamma\rangle$ in $\hat{\mathbb{C}}$, where Γ is a compact subset of Rat. In particular, it is important to analyze the reason of instability for a non-attracting minimal set.

Definition 3.10. For a $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ with $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$, let $\pi_\tau : C(\hat{\mathbb{C}}) \to \mathcal{U}_\tau$ be the canonical projection coming from Theorem A.

Theorem 3.11 (Theorem C, Stability). *Suppose $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is mean stable and $J(G_\tau) \neq \emptyset$. Then there exists a neighborhood Ω of τ in $(\mathfrak{M}_{1,c}(\mathrm{Rat}), \mathcal{O})$ such that all of the following* (a)(b)(c) *hold.*

(a) *For each $\nu \in \Omega$, ν is mean stable (thus* Theorem A *for ν holds).*

(b) *The maps $\nu \mapsto \pi_\nu$ and $\nu \mapsto \mathcal{U}_\nu$ are continuous on Ω. More precisely, for each $\nu \in \Omega$, there exists a family $\{\varphi_{j,\nu}\}_{j=1}^q$ of unitary eigenvectors of $M_\nu : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$, where $q = \dim_{\mathbb{C}}(\mathcal{U}_\tau)$, and a finite family $\{\rho_{j,\nu}\}_{j=1}^q$ in $C(\hat{\mathbb{C}})^* := \{\rho : C(\hat{\mathbb{C}}) \to \mathbb{C} \mid \rho$ is linear and continuous$\}$ (endowed with the weak*-topology) such that all of the following* (i)–(v) *hold.*

   (i) *$\{\varphi_{j,\nu}\}_{j=1}^q$ is a basis of $\mathcal{U}_\nu$.*

   (ii) *For each j, $\nu \mapsto \varphi_{j,\nu} \in C(\hat{\mathbb{C}})$ is continuous on Ω.*

   (iii) *For each j, $\nu \mapsto \rho_{j,\nu} \in C(\hat{\mathbb{C}})^*$ is continuous on Ω.*

   (iv) *For each (i,j) and each $\nu \in \Omega$, $\rho_{i,\nu}(\varphi_{j,\nu}) = \delta_{ij}$.*

   (v) *For each $\nu \in \Omega$ and each $\varphi \in C(\hat{\mathbb{C}})$, $\pi_\nu(\varphi) = \sum_{j=1}^q \rho_{j,\nu}(\varphi) \cdot \varphi_{j,\nu}$.*

(c) *The map $\nu \mapsto \sharp \mathrm{Min}(G_\nu, \hat{\mathbb{C}})$ is constant on Ω.*

By applying these results, we can give a characterization of mean stability (see [50]).
We now consider the speed of convergence of $M_\tau^n(\varphi - \pi_\tau(\varphi))$, where φ is a Hölder continuous function.

Definition 3.12. For each $\alpha \in (0,1)$, we set

$$C^\alpha(\hat{\mathbb{C}}) := \{\varphi \in C(\hat{\mathbb{C}}) \mid \|\varphi\|_\alpha < \infty\},$$

where $\|\varphi\|_\alpha := \sup_{z\in\hat{\mathbb{C}}} |\varphi(z)| + \sup_{x,y\in\hat{\mathbb{C}}, x\neq y} |\varphi(x) - \varphi(y)|/d(x,y)^\alpha$. ($\alpha$-Hölder norm.)

Theorem 3.13 (Theorem D, Exponential Rate of Convergence). *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose*

(1) *$J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, and*

(2) *for each minimal set L of G_τ in $\hat{\mathbb{C}}$, $L \subset F(G_\tau)$.*

(Note: if $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is mean stable and $J(G_\tau) \neq \emptyset$, then the above (1) and (2) hold.)
Then $\exists \alpha \in (0,1)$ $\exists C > 0$ $\exists \lambda \in (0,1)$ s.t.
for each α-Hölder continuous function φ on $\hat{\mathbb{C}}$ and for each $n \in \mathbb{N}$,

$$\|M_\tau^n(\varphi - \pi_\tau(\varphi))\|_\alpha \leq C\lambda^n \|\varphi\|_\alpha.$$

Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ be mean stable and suppose $J(G_\tau) \neq \emptyset$. Then by Theorem A, the chaos of the averaged system of τ disappears (cooperation principle), and by Theorem D, there exists an $\alpha_0 \in (0,1)$ such that for each $\alpha \in (0,1)$ the action of $\{M_\tau^n\}_{n\in\mathbb{N}}$ on $C^\alpha(\hat{\mathbb{C}})$ is well-behaved. However, Appendix (for more details, see [48, Theorem 3.82]) tells us that under certain conditions on a mean stable τ, there exists a $\beta \in (0,1)$ such that any non-constant element $\varphi \in \mathcal{U}_\tau$ does not belong to $C^\beta(\hat{\mathbb{C}})$ (note: for the proof of this result, we use the Birkhoff ergodic theorem and potential theory). Hence, there exists an element $\psi \in C^\beta(\hat{\mathbb{C}})$ such that $\|M_\tau^n(\psi)\|_\beta \to \infty$ as $n \to \infty$. Therefore, the action of $\{M_\tau^n\}_{n\in\mathbb{N}}$ on $C^\beta(\hat{\mathbb{C}})$ is not well behaved. In other words, regarding the dynamics of the averaged system of τ, there still exists a kind of chaos (or complexity) in the space $(C^\beta(\hat{\mathbb{C}}), \|\cdot\|_\beta)$ even though there exists no chaos in the space $(C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$. From this point of view, in the field of random dynamics, we have a kind of gradation or stratification between chaos and non-chaos.

It may be nice to investigate and reconsider the chaos theory and mathematical modeling from this point of view.

Under the assumptions of Theorem D, We now consider the spectrum $\mathrm{Spec}_\alpha(M_\tau)$ of $M_\tau : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})$. From Theorem D, denoting by $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ the set of unitary eigenvalues of $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ (note: by Theorem A, $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}}) \subset \mathrm{Spec}_\alpha(M_\tau)$ for some $\alpha \in (0,1)$), we can show that the distance between $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ and $\mathrm{Spec}_\alpha(M_\tau) \setminus \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ is positive.

Theorem 3.14. *Under the assumptions of Theorem D,* $\mathrm{Spec}_\alpha(M_\tau) \subset \{z \in \mathbb{C} \mid |z| \leq \lambda\} \cup \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$, *where* $\lambda \in (0,1)$ *denotes the constant in Theorem D.*

Combining Theorem 3.14 and perturbation theory for linear operators ([19]), we obtain the following theorem. We remark that even if $g_n \to g$ in Rat, for a $\varphi \in C^\alpha(\hat{\mathbb{C}})$, $\|\varphi \circ g_n - \varphi \circ g\|_\alpha$ does not tend to zero in general. Thus when we perturb generators $\{h_j\}$ of Γ_τ, we cannot apply perturbation theory for M_τ on $C^\alpha(\hat{\mathbb{C}})$. However, for a fixed generator system $(h_1, \dots h_m) \in \mathrm{Rat}^m$, the map $(p_1, \dots, p_m) \in \mathcal{W}_m := \{(a_1, \dots, a_m) \in (0,1)^m \mid \sum_{j=1}^m a_j = 1\} \mapsto M_{\sum_{j=1}^m p_j \delta_{h_j}} \in L(C^\alpha(\hat{\mathbb{C}}))$ is real-analytic, where $L(C^\alpha(\hat{\mathbb{C}}))$ denotes the Banach space of bounded linear operators on $C^\alpha(\hat{\mathbb{C}})$ endowed with the operator norm. Thus we can apply perturbation theory for the above real-analytic family of operators.

Theorem 3.15 (Theorem E: Complex Analogue of the Takagi Function). *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h_1, \dots, h_m \in$ Rat. Let $G = \langle h_1, \dots, h_m \rangle$. Suppose that $J_{\ker}(G) = \emptyset, J(G) \neq \emptyset$ and each minimal set L of G in $\hat{\mathbb{C}}$ is included in $F(G)$. Let $\mathcal{W}_m := \{(a_1, \dots, a_m) \in (0,1)^m \mid \sum_{j=1}^m a_j = 1\} \cong \{(a_1, \dots, a_{m-1}) \in (0,1)^{m-1} \mid \sum_{j=1}^{m-1} a_j < 1\}$. For each $a = (a_1, \dots, a_m) \in W_m$, let $\tau_a := \sum_{j=1}^m a_j \delta_{h_j} \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Then we have all of the following.*

1. *For each $b \in \mathcal{W}_m$, there exists an $\alpha \in (0,1)$ such that $a \mapsto (\pi_{\tau_a} : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})) \in L(C^\alpha(\hat{\mathbb{C}}))$, where $L(C^\alpha(\hat{\mathbb{C}}))$ denotes the Banach space of bounded linear operators on $C^\alpha(\hat{\mathbb{C}})$ endowed with the operator norm, is real-analytic in an open neighborhood of b in $\mathcal{W}_m$.*
2. *Let L be a minimal set of G in $\hat{\mathbb{C}}$. Then, for each $b \in \mathcal{W}_m$, there exists an $\alpha \in (0,1)$ such that the map $a \mapsto T_{L,\tau_a} \in (C^\alpha(\hat{\mathbb{C}}), \|\cdot\|_\alpha)$ is real-analytic in an open neighborhood of b in $\mathcal{W}_m$. Moreover, the map $a \mapsto T_{L,\tau_a} \in (C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$ is real-analytic in $\mathcal{W}_m$. In particular, for each $z \in \hat{\mathbb{C}}$, the map $a \mapsto T_{L,\tau_a}(z)$ is real-analytic in $\mathcal{W}_m$. Furthermore, for any multi-index $n = (n_1, \dots, n_{m-1}) \in (\mathbb{N} \cup \{0\})^{m-1}$ and for any $b \in \mathcal{W}_m$, the function $z \mapsto [(\frac{\partial}{\partial a_1})^{n_1} \cdots (\frac{\partial}{\partial a_{m-1}})^{n_{m-1}} (T_{L,\tau_a}(z))]|_{a=b}$ is Hölder continuous on $\hat{\mathbb{C}}$ and is locally constant on $F(G)$.*
3. *Let L be a minimal set of G in $\hat{\mathbb{C}}$ and let $b \in \mathcal{W}_m$. For each $i = 1, \dots, m-1$ and for each $z \in \hat{\mathbb{C}}$, let $\psi_{i,b,L}(z) := [\frac{\partial}{\partial a_i}(T_{L,\tau_a}(z))]|_{a=b}$ and let $\zeta_{i,b,L}(z) := T_{L,\tau_b}(h_i(z)) - T_{L,\tau_b}(h_m(z))$. Then, $\psi_{i,b,L}$ is the unique solution of the functional equation $(I - M_{\tau_b})(\psi) = \zeta_{i,b,L}, \psi|_{S_{\tau_b}} = 0, \psi \in C(\hat{\mathbb{C}})$, where I denotes the identity map. Moreover, there exists a number $\alpha \in (0,1)$ such that $\psi_{i,b,L} = \sum_{n=0}^\infty M_{\tau_b}^n(\zeta_{i,b,L})$ in $(C^\alpha(\hat{\mathbb{C}}), \|\cdot\|_\alpha)$.*

The function $T_{L,\tau}$ is a complex analogue of the devil's staircase or Lebesgue's singular functions, and the function $\psi_{i,b,L}$ is a complex analogue of the Takagi function. We can investigate the pointwise Hölder exponents and non-differentiability of $T_{L,\tau}$ and $\psi_{i,b,L}$ at points in $J_{\ker}(G_\tau)$, by using ergodic theory, potential theory, and function theory. See Appendix and [48, 50].

We now present a result on bifurcation.

Theorem 3.16 (Bifurcation). *Let $\mathcal{Y}$ be a subset of Rat_+ satisfying condition $(*)$. For each $t \in [0,1]$, let μ_t be an element of $\mathfrak{M}_{1,c}(\mathcal{Y})$. Suppose that all of the following (1)–(4) hold.*

(1) $t \mapsto \mu_t \in (\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ *is continuous on* $[0,1]$.

(2) *If* $t_1, t_2 \in [0,1]$ *and* $t_1 < t_2$*, then* $\operatorname{supp}\mu_{t_1} \subset \operatorname{int}(\operatorname{supp}\mu_{t_2})$ *with respect to the topology of* $\mathcal{Y}$.

(3) $\operatorname{int}(\operatorname{supp}\mu_0) \neq \emptyset$ *and* $F(G_{\mu_1}) \neq \emptyset$.

(4) $\sharp(\operatorname{Min}(G_{\mu_0}, \hat{\mathbb{C}})) \neq \sharp(\operatorname{Min}(G_{\mu_1}, \hat{\mathbb{C}}))$.

Let $B := \{t \in [0,1) \mid \mu_t$ *is not mean stable*$\}$*. Then, we have the following.*

(a) *For each* $t \in [0,1]$*,* $J_{\ker}(G_{\mu_t}) = \emptyset$ *and* $\sharp J(G_{\mu_t}) \geq 3$*, and all statements in Theorem A (with* $\tau = \mu_t$*) hold.*

(b) *We have* $1 \leq \sharp B \leq \sharp\operatorname{Min}(G_{\mu_0}, \hat{\mathbb{C}}) - \sharp\operatorname{Min}(G_{\mu_1}, \hat{\mathbb{C}}) < \infty$*. Moreover, for each* $t \in B$*, either* (i) *there exists an element* $L \in \operatorname{Min}(G_{\mu_t}, \hat{\mathbb{C}})$*, a point* $z \in L$*, and an element* $g \in \partial\Gamma_{\mu_t} (\subset \mathcal{Y})$ *such that* $z \in L \cap J(G_{\mu_t})$ *and* $g(z) \in L \cap J(G_{\mu_t})$*, or* (ii) *there exist an element* $L \in \operatorname{Min}(G_{\mu_t}, \hat{\mathbb{C}})$*, a point* $z \in L$*, and finitely many elements* $g_1, \ldots, g_r \in \partial\Gamma_{\mu_t}$ *such that* $L \subset F(G_{\mu_t})$ *and* z *belongs to a Siegel disk or a Hermann ring of* $g_r \circ \cdots \circ g_1$.

To give an example which describes the above theorem, let c_0 be a point in the interior of the Mandelbrot set $\mathcal{M} := \{c \in \mathbb{C} \mid \{h_c^n(c)\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{C}\}$, where $h_c(z) := z^2 + c$. Suppose h_{c_0} is hyperbolic (i.e., $\overline{\bigcup_{n\in\mathbb{N}}\{h_c^n(c)\}} \subset F(h_c)$). Let $r_0 > 0$ be a small number. Let $r_1 > 0$ be a large number such that $D(c_0, r_1) \cap (\mathbb{C} \setminus \mathcal{M}) \neq \emptyset$. For each $t \in [0,1]$, let $\mu_t \in \mathfrak{M}_1(\overline{D(c_0, (1-t)r_0 + tr_1)})$ be the normalized 2-dimensional Lebesgue measure on $\overline{D(c_0, (1-t)r_0 + tr_1)}$. Then $\{\mu_t\}_{t\in[0,1]}$ satisfies the conditions (1)–(4) in Theorem 3.16 (for example, $2 = \sharp\operatorname{Min}(G_{\mu_0}, \hat{\mathbb{C}}) > \sharp\operatorname{Min}(G_{\mu_1}, \hat{\mathbb{C}}) = 1$). Thus

$$\sharp\{t \in [0,1] \mid \mu_t \text{ is not mean stable}\} = 1.$$

4 Examples

Example 4.1 (Devil's coliseum ([48]) and complex analogue of the Takagi function)**.** Let $g_1(z) := z^2 - 1, g_2(z) := z^2/4, h_1 := g_1^2$, and $h_2 := g_2^2$. Let $G = \langle h_1, h_2 \rangle$ and for each $a = (a_1, a_2) \in \mathcal{W}_2 := \{(a_1, a_2) \in (0,1)^2 \mid \sum_{j=1}^2 a_j = 1\} \cong (0,1)$, let $\tau_a := \sum_{i=1}^2 a_i \delta_{h_i}$. Then by [48, Example 6.2], $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$. Moreover, G is hyperbolic (see Definition 6.1). Moreover, we can show that for each $a \in \mathcal{W}_2$, τ_a is mean stable, T_{∞,τ_a} is continuous on $\hat{\mathbb{C}}$, and the set of varying points of T_{∞,τ_a} is equal to $J(G)$. Moreover, by [48] $\dim_H(J(G)) < 2$ and for each non-empty open subset U of $J(G)$ there exists an uncountable dense subset A_U of U such that for each $z \in A_U$, T_{∞,τ_a} is not differentiable at z. See Figures 2 and 3. T_{∞,τ_a} is called a devil's coliseum. It is a complex analogue of the devil's staircase (see Introduction). By Theorem E, for each $z \in \hat{\mathbb{C}}$, $a_1 \mapsto T_{\infty,\tau_a}(z)$ is real-analytic in $(0,1)$, and for each $b \in \mathcal{W}_2$, $[\frac{\partial T_{\infty,\tau_a}(z)}{\partial a_1}]|_{a=b} = \sum_{n=0}^{\infty} M_{\tau_b}^n(\zeta_{1,b})$, where $\zeta_{1,b}(z) := T_{\infty,\tau_b}(h_1(z)) - T_{\infty,\tau_b}(h_2(z))$. Moreover, by Theorem E, the function $\psi(z) := [\frac{\partial T_{\infty,\tau_a}(z)}{\partial a_1}]|_{a=b}$ defined on $\hat{\mathbb{C}}$ is Hölder continuous on $\hat{\mathbb{C}}$ and is locally constant on $F(G)$. The function $\psi(z)$ defined on $\hat{\mathbb{C}}$ can be regarded as a complex analogue of the Takagi function (see Introduction). We can show that there exists an uncountable dense subset A of $J(G)$ such that for each $z \in A$, either ψ is not differentiable at z or ψ is not differentiable at each point $w \in h_1^{-1}(\{z\}) \cup h_2^{-1}(\{z\})$ (see Appendix). For the graph of $[\frac{\partial T_{\infty,\tau_a}(z)}{\partial a_1}]|_{a_1=1/2}$, see Figure 5.

Figure 2: The Julia set of $G = \langle h_1, h_2 \rangle$, where $g_1(z) := z^2 - 1, g_2(z) := z^2/4, h_1 := g_1^2, h_2 := g_2^2$. $P^*(G)$ (see Definition 6.1) is bounded in $\mathbb{C}$ and $J(G)$ has uncountably many connected components. G is hyperbolic ([47]). $\bigcap_{i=1}^2 h_i^{-1}(J(G)) = \emptyset$ and (h_1, h_2) satisfies the open set condition ([52]). Moreover, for each connected component J of $J(G)$, there exists a unique $\gamma \in \{h_1, h_2\}^{\mathbb{N}}$ such that $J = J_\gamma$. For almost every $\gamma \in \{h_1, h_2\}^{\mathbb{N}}$ with respect to a Bernoulli measure, J_γ is a simple closed curve but not a quasicircle, and the basin A_γ of infinity for the sequence γ is a John domain ([47]).

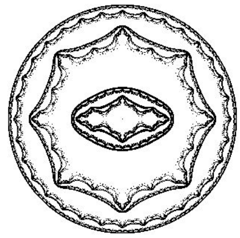


Figure 3: The graph of $z \mapsto T_{\infty, \tau_{1/2}}(z)$, where, letting (h_1, h_2) be the element in Figure 2, we set $\tau_a := \sum_{j=1}^2 a_j \delta_{h_j}$. A devil's coliseum (a complex analogue of the devil's staircase). τ_a is mean stable. The set of varying points is equal to Figure 2.

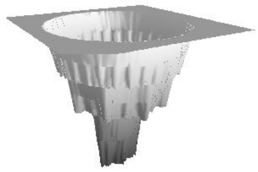


Figure 4: Figure 3 upside down.

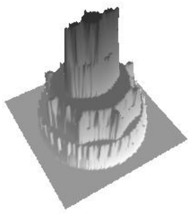
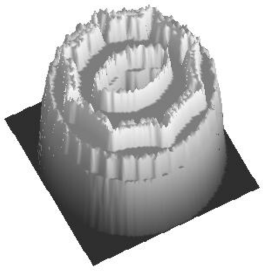


Figure 5: The graph of $z \mapsto [(\partial T_{\infty, \tau_a}(z)/\partial a_1)]|_{a_1=1/2}$, where, τ_a is the element in Figure 3. A complex analogue of the Takagi function.



5 Summary

In the random complex dynamics of polynomials, for a generic probability measure τ on the space of (polynomial) maps,

- the chaos of the averaged system disappears, due to the **automatic cooperation** of the generator maps (even though each map of the system has a chaotic part),
- there exists a stability of the limit state w.r.t. the perturbation, and
- the orbit of a Hölder continuous function under the transition operator M_τ converges exponentially fast to the finite-dimensional space $\mathcal{U}_\tau$ of finite linear combinations of unitary eigenvectors of M_τ.

6 Appendix: pointwise Hölder exponent and (non-)differentiability of $T_{L,\tau}$ and $\psi_{i,b,L}$ at points in $J(G_\tau)$

In this appendix, we consider the pointwise Hölder exponent and (non-)differentiability of $T_{L,\tau}$ (a complex analogue of the devil's staircase and Lebesgue's singular functions) and $\psi_{i,b,L}$ (partial derivative of $T_{L,\tau}$ with respect to a probability parameter: a complex analogue of the Takagi function) at points in $J(G_\tau)$. We use ergodic theory, potential theory, and function theory.

Definition 6.1. For a rational semigroup G, we set $P(G) := \overline{\bigcup_{g\in G}\{\text{ all critical values of } g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}}$ where the closure is taken in $\hat{\mathbb{C}}$. This is called the postcritical set of G. We say that a rational semigroup G is hyperbolic if $P(G) \subset F(G)$. For a polynomial semigroup G, we set $P^*(G) := P(G)\setminus\{\infty\}$. For a polynomial semigroup G, we set $\hat{K}(G) := \{z \in \mathbb{C} \mid \bigcup_{g\in G}\{g(z)\} \text{ is bounded in } \mathbb{C}\}$. Moreover, for each polynomial h, we set $K(h) := \hat{K}(\langle h\rangle)$. For a topological space X, we denote by $\mathrm{Cpt}(X)$ the space of all non-empty compact subsets of X. If X is a metric space, we endow $\mathrm{Cpt}(X)$ with the Hausdorff metric.

Remark 6.2. Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ and suppose that $\langle\Gamma\rangle$ is hyperbolic and $J_{\ker}(\langle\Gamma\rangle) = \emptyset$. Then by [48, Propositions 3.63, 3.65], there exists an neighborhood $\mathcal{U}$ of Γ in $\mathrm{Cpt}(\mathrm{Rat})$ such that for each $\tau \in \mathcal{M}_{1,c}(\mathrm{Rat})$ with $\mathrm{supp}\,\tau \in \mathcal{U}$, τ is mean stable, $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$ and $\bigcup_{L\in\mathrm{Min}(G_\tau,\hat{\mathbb{C}})} L \subset F(G_\tau)$.

Definition 6.3. Let $m \in \mathbb{N}$. Let $h = (h_1, \ldots, h_m) \in (\mathrm{Rat})^m$ be an element such that $h_1, \ldots, h_m$ are mutually distinct. We set $\Gamma := \{h_1, \ldots, h_m\}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the map defined by $f(\gamma, y) = (\sigma(\gamma), \gamma_1(y))$, where $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^{\mathbb{N}}$ and $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ is the shift map $((\gamma_1, \gamma_2, \ldots) \mapsto (\gamma_2, \gamma_3, \ldots))$. This map $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ is called the skew product associated with Γ. Let $\pi : \Gamma^{\mathbb{N}}\times\hat{\mathbb{C}} \to \Gamma^{\mathbb{N}}$ and $\pi_{\hat{\mathbb{C}}} : \Gamma^{\mathbb{N}}\times\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the canonical projections. Let $\mu \in \mathfrak{M}_1(\Gamma^{\mathbb{N}}\times\hat{\mathbb{C}})$ be an f-invariant Borel probability measure. Let $\mathcal{W}_m := \{(a_1, \ldots, a_m) \in (0, 1)^m \mid \sum_{j=1}^m a_j = 1\}$. For each $p = (p_1, \ldots, p_m) \in \mathcal{W}_m$, we define a function $\tilde{p} : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \mathbb{R}$ by $\tilde{p}(\gamma, y) := p_j$ if $\gamma_1 = h_j$ (where $\gamma = (\gamma_1, \gamma_2, \ldots)$), and we set

$$u(h, p, \mu) := \frac{-(\int_{\Gamma^{\mathbb{N}}\times\hat{\mathbb{C}}} \log\tilde{p}(\gamma, y)\ d\mu(\gamma, y))}{\int_{\Gamma^{\mathbb{N}}\times\hat{\mathbb{C}}} \log\|(D\gamma_1)_y\|_s\ d\mu(\gamma, y)}$$

(when the integral of the denominator converges), where $\|\cdot\|_s$ denotes the norm of the derivative with respect to the spherical metric on $\hat{\mathbb{C}}$.

Definition 6.4. Let $h = (h_1, \ldots, h_m) \in \mathcal{P}^m$ be an element such that $h_1, \ldots, h_m$ are mutually distinct. We set $\Gamma := \{h_1, \ldots, h_m\}$. For any $(\gamma, y) \in \Gamma^{\mathbb{N}}\times\mathbb{C}$, let $\mathcal{G}_\gamma(y) := \lim_{n\to\infty} \frac{1}{\deg(\gamma_{n,1})} \log^+ |\gamma_{n,1}(y)|$, where $\log^+ a := \max\{\log a, 0\}$ for each $a > 0$. By the arguments in [25], for each $\gamma \in \Gamma^{\mathbb{N}}$, $\mathcal{G}_\gamma(y)$ exists, $\mathcal{G}_\gamma$ is subharmonic on $\mathbb{C}$, and $\mathcal{G}_\gamma|_{A_{\infty,\gamma}}$ is equal to the Green's function on $A_{\infty,\gamma}$ with pole at ∞, where $A_{\infty,\gamma} := \{z \in \hat{\mathbb{C}} \mid \gamma_{n,1}(z) \to \infty \text{ as } n \to \infty\}$. Moreover, $(\gamma, y) \mapsto \mathcal{G}_\gamma(y)$ is continuous on $\Gamma^{\mathbb{N}} \times \mathbb{C}$. Let $\mu_\gamma := dd^c \mathcal{G}_\gamma$, where $d^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$. Note that by the argument in [17, 25], μ_γ is a Borel probability measure on J_γ such that $\operatorname{supp} \mu_\gamma = J_\gamma$. Furthermore, for each $\gamma \in \Gamma^{\mathbb{N}}$, let $\Omega(\gamma) = \sum_c \mathcal{G}_\gamma(c)$, where c runs over all critical points of γ_1 in $\mathbb{C}$, counting multiplicities.

Remark 6.5. Let $h = (h_1, \dots, h_m) \in (\mathrm{Rat}_+)^m$ be an element such that $h_1, \dots, h_m$ are mutually distinct. Let $\Gamma = \{h_1, \dots, h_m\}$ and let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product map associated with Γ. Moreover, let $p = (p_1, \dots, p_m) \in \mathcal{W}_m$ and let $\tau = \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma)$. Then, there exists a unique f-invariant ergodic Borel probability measure μ on $\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ such that $\pi_*(\mu) = \tilde{\tau}$ and $h_\mu(f|\sigma) = \max_{\rho \in \mathfrak{E}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}) : f_*(\rho) = \rho, \pi_*(\rho) = \tilde{\tau}} h_\rho(f|\sigma) = \sum_{j=1}^m p_j \log(\deg(h_j))$, where $h_\rho(f|\sigma)$ denotes the relative metric entropy of (f, ρ) with respect to $(\sigma, \tilde{\tau})$, and $\mathfrak{E}_1(\cdot)$ denotes the space of ergodic measures for f (see [36]). This μ is called the **maximal relative entropy measure** for f with respect to $(\sigma, \tilde{\tau})$.

Definition 6.6. Let V be a non-empty open subset of $\hat{\mathbb{C}}$. Let $\varphi : V \to \mathbb{C}$ be a function and let $y \in V$ be a point. Suppose that φ is bounded around y. Then we set

$$\mathrm{H\ddot{o}l}(\varphi, y) := \inf\{\beta \in \mathbb{R} \mid \limsup_{z \to y} \frac{|\varphi(z) - \varphi(y)|}{d(z, y)^\beta} = \infty\},$$

where d denotes the spherical distance. This is called the **pointwise Hölder exponent of φ at y**.

Remark 6.7. If $\mathrm{H\ddot{o}l}(\varphi, y) < 1$, then φ is non-differentiable at y. If $\mathrm{H\ddot{o}l}(\varphi, y) > 1$, then φ is differentiable at y and the derivative at y is equal to 0.

We now present a result on the non-differentiability of T_{L,τ_a} and $\psi_{i,b,L}(z) = [\frac{\partial}{\partial a_i}(T_{L,\tau_a}(z))]|_{a=b}$ (in Theorem E) at points in $J(G_\tau)$.

Theorem 6.8. *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \dots, h_m) \in (\mathrm{Rat}_+)^m$ and we set $\Gamma := \{h_1, h_2, \dots, h_m\}$. Let $G = \langle h_1, \dots, h_m \rangle$. Let $\mathcal{W}_m := \{(a_1, \dots, a_m) \in (0,1)^m \mid \sum_{j=1}^m a_j = 1\} \cong \{(a_1, \dots, a_{m-1}) \in (0,1)^{m-1} \mid \sum_{j=1}^{m-1} a_j < 1\}$. For each $a = (a_1, \dots, a_m) \in \mathcal{W}_m$, let $\tau_a := \sum_{j=1}^m a_j \delta_{h_j} \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Let $p = (p_1, \dots, p_m) \in \mathcal{W}_m$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let $\tau := \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma) \subset \mathfrak{M}_1(\mathcal{P})$. Let $\mu \in \mathfrak{M}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}})$ be the maximal relative entropy measure for $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ with respect to $(\sigma, \tilde{\tau})$. Moreover, let $\lambda := (\pi_{\hat{\mathbb{C}}})_*(\mu) \in \mathfrak{M}_1(\hat{\mathbb{C}})$. Suppose that G is hyperbolic, and $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i, j) with $i \neq j$. For each $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$, for each $i = 1, \dots, m-1$ and for each $z \in \hat{\mathbb{C}}$, let $\psi_{i,p,L}(z) := [\frac{\partial}{\partial a_i}(T_{L,\tau_a}(z))]|_{a=p}$. Then, we have all of the following.*

1. *τ is mean stable, $J_{\ker}(G) = \emptyset$, and $S_\tau \subset F(G_\tau)$. Moreover, $0 < \dim_H(J(G)) < 2$, supp $\lambda = J(G)$, and $\lambda(\{z\}) = 0$ for each $z \in J(G)$.*
2. *There exists a Borel subset A of $J(G)$ with $\lambda(A) = 1$ such that for each $z_0 \in A$ and for each non-constant $\varphi \in \mathcal{U}_\tau$, $\mathrm{H\ddot{o}l}(\varphi, z_0) = u(h, p, \mu)$.*
3. *Suppose $\sharp \mathrm{Min}(G, \hat{\mathbb{C}}) \neq 1$. Then there exists a Borel subset A of $J(G)$ with $\lambda(A) = 1$ such that for each $z_0 \in A$, for each $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ and for each $i = 1, \dots, m-1$, exactly one of the following* (a),(b),(c) *holds.*
   (a) $\mathrm{H\ddot{o}l}(\psi_{i,p,L}, z_1) = \mathrm{H\ddot{o}l}(\psi_{i,p,L}, z_0) < u(h, p, \mu)$ *for each* $z_1 \in h_i^{-1}(\{z_0\}) \cup h_m^{-1}(\{z_0\})$.
   (b) $\mathrm{H\ddot{o}l}(\psi_{i,p,L}, z_0) = u(h, p, \mu) \leq \mathrm{H\ddot{o}l}(\psi_{i,p,L}, z_1)$ *for each* $z_1 \in h_i^{-1}(\{z_0\}) \cup h_m^{-1}(\{z_0\})$.

(c) $\mathrm{H\ddot{o}l}(\psi_{i,p,L}, z_1) = u(h,p,\mu) < \mathrm{H\ddot{o}l}(\psi_{i,p,L}, z_0)$ *for each* $z_1 \in h_i^{-1}(\{z_0\}) \cup h_m^{-1}(\{z_0\})$.

4. *If* $h = (h_1, \ldots, h_m) \in \mathcal{P}^m$, *then*

$$u(h,p,\mu) = \frac{-(\sum_{j=1}^m p_j \log p_j)}{\sum_{j=1}^m p_j \log \deg(h_j) + \int_{\Gamma^{\mathbb{N}}} \Omega(\gamma)\ d\tilde{\tau}(\gamma)}$$

and

$$2 > \dim_H(\lambda) = \frac{\sum_{j=1}^m p_j \log \deg(h_j) - \sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \log \deg(h_j) + \int_{\Gamma^{\mathbb{N}}} \Omega(\gamma)\ d\tilde{\tau}(\gamma)} > 0.$$

5. *Suppose* $h = (h_1, \ldots, h_m) \in \mathcal{P}^m$. *Moreover, suppose that at least one of the following* (a), (b), *and* (c) *holds:* (a) $\sum_{j=1}^m p_j \log(p_j \deg(h_j)) > 0$. (b) $P^*(G)$ *is bounded in* $\mathbb{C}$. (c) $m = 2$. *Then,* $u(h,p,\mu) < 1$ *and for each non-empty open subset* U *of* $J(G)$ *there exists an uncountable dense subset* A_U *of* U *such that for each* $z \in A_U$ *and for each non-constant* $\varphi \in \mathcal{U}_\tau$, φ *is non-differentiable at* z.

Remark 6.9. By Theorems A and 6.8, it follows that under the assumptions of Theorem 6.8, the chaos of the averaged system disappears in the C^0 "sense", but it remains in the C^1 "sense".

We now present a result on the representation of pointwise Hölder exponent of non-constant $\varphi \in \mathcal{U}_\tau$ at almost every point in $J(G_\tau)$ with respect to the δ-dimensional Hausdorff measure, where $\delta = \dim_H(J(G_\tau))$.

Definition 6.10. Let $m \in \mathbb{N}$ and let $(h_1, \ldots, h_m) \in (\mathrm{Rat})^m$. Let $\Gamma = \{h_1, \ldots, h_m\}$. Let $\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^{\mathbb{N}}$, we set $J_\gamma := \{z \in \hat{\mathbb{C}} \mid \forall \text{ nbd } U \text{ of } z, \{\gamma_n \cdots \gamma_1|_U : U \to \hat{\mathbb{C}}\}_{n\in\mathbb{N}} \text{ is not equiconti. on } U\}$. Let $\tilde{J}(f) := \overline{\bigcup_{\gamma\in\Gamma^{\mathbb{N}}} \{\gamma\} \times J_\gamma}$, where the closure is taken in the product space $\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$.

For each compact metric space X, we set $C(X) := \{\varphi : X \to \mathbb{C} \mid \varphi \text{ is conti.}\}$ endowed with the supremum norm.

Theorem 6.11. *Let* $m \in \mathbb{N}$ *with* $m \geq 2$. *Let* $h = (h_1, \ldots, h_m) \in (\mathrm{Rat}_+)^m$ *and we set* $\Gamma := \{h_1, h_2, \ldots, h_m\}$. *Let* $G = \langle h_1, \ldots, h_m \rangle$. *Let* $p = (p_1, \ldots, p_m) \in \mathcal{W}_m$. *Let* $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ *be the skew product associated with* Γ. *Let* $\tau := \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma) \subset \mathfrak{M}_1(\mathrm{Rat}_+)$. *Suppose that* G *is hyperbolic and* $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ *for each* (i,j) *with* $i \neq j$. *Let* $\delta := \dim_H(J(G))$ *and let* H^δ *be the* δ*-dimensional Hausdorff measure. Let* $\tilde{L} : C(\tilde{J}(f)) \to C(\tilde{J}(f))$ *be the operator defined by* $\tilde{L}(\varphi)(z) = \sum_{f(\gamma,w)=z} \varphi(\gamma, w) \|(D\gamma_1)_w\|_s^{-\delta}$, *where* $\gamma = (\gamma_1, \gamma_2, \ldots)$, *and* $\|\cdot\|_s$ *denotes the norm of the derivative with respect to the spherical metric on* $\hat{\mathbb{C}}$. *Moreover, let* $L : C(J(G)) \to C(J(G))$ *be the operator defined by* $L(\varphi)(z) = \sum_{j=1}^m \sum_{h_j(w)=z} \varphi(w) \|(Dh_j)_w\|_s^{-\delta}$. *Then, we have all of the following.*

1. τ *is mean stable and* $J_{\ker}(G) = \emptyset$.

2. *There exists a unique element* $\tilde{\nu} \in \mathfrak{M}_1(\tilde{J}(f))$ *such that* $\tilde{L}^*(\tilde{\nu}) = \tilde{\nu}$. *Moreover, the limits* $\tilde{\alpha} = \lim_{n\to\infty} \tilde{L}^n(1) \in C(\tilde{J}(f))$ *and* $\alpha = \lim_{n\to\infty} L^n(1) \in C(J(G))$ *exist, where* 1 *denotes the constant function taking its value* 1.

3. *Let* $\nu := (\pi_{\hat{\mathbb{C}}})_*(\tilde{\nu}) \in \mathfrak{M}_1(J(G))$. *Then* $0 < \delta < 2$, $0 < H^\delta(J(G)) < \infty$, *and* $\nu = \frac{H^\delta}{H^\delta(J(G))}$.

4. *Let* $\tilde{\rho} := \tilde{\alpha}\tilde{\nu} \in \mathfrak{M}_1(\tilde{J}(f))$. *Then* $\tilde{\rho}$ *is* f*-invariant and ergodic. Moreover,* $\min_{z\in J(G)} \alpha(z) > 0$.

5. *There exists a Borel subset of* A *of* $J(G)$ *with* $H^\delta(A) = H^\delta(J(G))$ *such that for each* $z_0 \in A$ *and for each non-constant* $\varphi \in \mathcal{U}_\tau$,

$$\mathrm{H\ddot{o}l}(\varphi, z_0) = u(h,p,\tilde{\rho}) = \frac{-\sum_{j=1}^m (\log p_j) \int_{h_j^{-1}(J(G))} \alpha(y)\ dH^\delta(y)}{\sum_{j=1}^m \int_{h_j^{-1}(J(G))} \alpha(y) \log \|(Dh_j)_y\|_s\ dH^\delta(y)}.$$

Remark 6.12. Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \ldots, h_m) \in \mathcal{P}^m$ and let $G = \langle h_1, \ldots, h_m \rangle$. Let $p = (p_1, \ldots, p_m) \in \mathcal{W}_m$ and let $\tau = \sum_{j=1}^m p_j \delta_{h_j}$. Suppose that $\hat{K}(G) \neq \emptyset$, G is hyperbolic, and $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i,j) with $i \neq j$. Then, $T_{\infty,\tau}$ belongs to $\mathcal{U}_\tau$ and is non-constant.

Remark 6.13. Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \ldots, h_m) \in \mathcal{P}^m$ and we set $\Gamma := \{h_1, \ldots, h_m\}$. Let $G = \langle h_1, \ldots, h_m \rangle$. Let $p = (p_1, \ldots, p_m) \in \mathcal{W}_m$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let $\tau := \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma) \subset \mathfrak{M}_1(\mathcal{P})$. Suppose that $\hat{K}(G) \neq \emptyset$, G is hyperbolic, and $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i,j) with $i \neq j$. Moreover, suppose we have at least one of the following (a),(b),(c): (a) $\sum_{j=1}^m p_j \log(p_j \deg(h_j)) > 0$. (b) $P^*(G)$ is bounded in $\mathbb{C}$. (c) $m = 2$. Then, combining Theorem 6.8, Theorem 6.11, and Remark 6.12, it follows that there exists a number $q > 0$ such that if $p_1 < q$, then we have all of the following.

1. Let μ be the maximal relative entropy measure for f with respect to $(\sigma, \tilde{\tau})$. Let $\lambda = (\pi_{\hat{\mathbb{C}}})_* \mu \in \mathfrak{M}_1(J(G))$. Then for λ-a.e. $z_0 \in J(G)$ and for each non-constant $\varphi \in \mathcal{U}_\tau$ (e.g., $\varphi = T_{\infty,\tau}$), $\limsup_{y \to z_0} \frac{|\varphi(y) - \varphi(z_0)|}{|y - z_0|} = \infty$ and φ is not differentiable at z_0.
2. Let $\delta = \dim_H(J(G))$ and let H^δ be the δ-dimensional Hausdorff measure. Then $0 < H^\delta(J(G)) < \infty$ and for H^δ-a.e. $z_0 \in J(G)$ and for any $\varphi \in \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$ (e.g., $\varphi = T_{\infty,\tau}$), $\limsup_{y \to z_0} \frac{|\varphi(y) - \varphi(z_0)|}{|y - z_0|} = 0$ and φ is differentiable at z_0.

Combining Theorem 3.1 and Theorem 6.8, we obtain the following result.

Corollary 6.14. *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \ldots, h_m) \in \mathcal{P}^m$ and we set $\Gamma := \{h_1, \ldots, h_m\}$. Let $G = \langle h_1, \ldots, h_m \rangle$. Let $p = (p_1, \ldots, p_m) \in \mathcal{W}_m$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let $\tau := \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma) \subset \mathfrak{M}_1(\mathcal{P})$. Suppose that $\hat{K}(G) \neq \emptyset$, G is hyperbolic, and $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i,j) with $i \neq j$. Moreover, suppose we have at least one of the following* (a), (b), (c): (a) $\sum_{j=1}^m p_j \log(p_j \deg(h_j)) > 0$. (b) *$P^*(G)$ is bounded in $\mathbb{C}$.* (c) $m = 2$. *Let $\varphi \in C(\hat{\mathbb{C}})$. Then, we have exactly one of the following* (i) *and* (ii).

(i) *There exists a constant function $\zeta \in C(\hat{\mathbb{C}})$ such that $M_\tau^n(\varphi) \to \zeta$ as $n \to \infty$ in $C(\hat{\mathbb{C}})$.*

(ii) *There exists a non-constant element $\psi \in \mathcal{U}_\tau$ and a number $l \in \mathbb{N}$ such that*

- $M_\tau^l(\psi) = \psi$,
- *each element of $\{M_\tau^j(\psi)\}_{j=0}^{l-1}$ belongs to $\mathcal{U}_\tau$, is non-constant, and is locally constant on $F(G)$,*
- *there exists an uncountable dense subset A of $J(G)$ such that for each $z_0 \in A$ and for each j, $M_\tau^j(\psi)$ is not differentiable at z_0, and*
- $M_\tau^{nl+j}(\varphi) \to M_\tau^j(\psi)$ *as $n \to \infty$ for each $j = 0, \ldots, l-1$.*

Remark 6.15. In the proof of Theorem 6.8, we use the Birkhoff ergodic theorem and the Koebe distortion theorem, in order to show that for each non-constant $\varphi \in \mathcal{U}_\tau$, $\mathrm{H\ddot{o}l}(\varphi, z_0) = u(h, p, \mu)$. Moreover, we apply potential theory in order to calculate $u(h, p, \mu)$ by using p, $\deg(h_j)$, and $\Omega(\gamma)$.

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