TROPICAL NEVANLINNA THEORY IN A SINGLE VARIABLE

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Abstract. Tropical Nevanlinna theory describes value distribution of continuous piecewise linear functions of a real variable with arbitrary real slopes, called tropical meromorphic functions, similarly as meromorphic functions are described in the classical Nevanlinna theory. In two previous papers, due to Halburd and Southall, resp. to Laine and Yang, integer slopes only had been permitted. We show that basic results of tropical Nevanlinna theory given in these two papers continue to be valid in the extended setting as well. In this talk, we present a tropical version of the second main theorem reminiscent to the corresponding result in the classical Nevanlinna theory. If time permits, we try to observe a possibility for Tropical Nevanlinna theory in an open interval on the real line, which corresponds to the classical Nevanlinna theory in an annulus or a disk, or else on the whole plane.

1. Introduction

Tropical Nevanlinna theory, see [7], describes value distribution of continuous piecewise linear functions of a real variable whose one-sided derivatives are integers at every point, similarly as meromorphic functions are described in the classical Nevanlinna theory [1], [8], [10]. In this talk, following [11], we take an extended point of view to tropical meromorphic functions by dispensing with the requirement of integer one-sided derivatives. Accepting that multiplicities of poles, resp. roots, may be arbitrary real numbers instead of being integers, resp. rationals, as in the classical theory of (complex) meromorphic functions, resp. of algebroid functions, it appears that previous results such as in [7], [12], continue to be valid, with slight modifications only in the proofs.

Recalling the standard one-dimensional tropical framework, we shall consider a max-plus semi-ring endowing $\mathbb{R}_{-\infty} := \mathbb{R} \cup \{-\infty\}$ with tropical addition

$$x \oplus y := \max(x, y)$$

and tropical multiplication

$$x \otimes y := x + y.$$

We also use the notations $x \oslash y := x - y$ and $x^{\otimes \alpha} := \alpha x$, for $\alpha \in \mathbb{R}$. The neutral elements for the tropical operations are $0_\circ = -\infty$ for addition and $1_\circ = 0$ for multiplication. We note that $1_\circ \oplus x = \max(x, 0)$ and $0_\circ \otimes x = 0_\circ$ for any $x \in \mathbb{R}$. Observe that such a structure is not a ring, since not all elements have tropical additive inverses. For a general background concerning tropical mathematics, see [15], for example.

Concerning meromorphic functions in the tropical setting, and their elementary Nevanlinna theory, see the recent paper by Halburd and Southall [7] as well as [12] and [11] for certain additional developments.

Definition 1.1 ([11]). A continuous piecewise linear function : $\mathbb{R} \to \mathbb{R}$ is said to be tropical meromorphic.

Remarks. (1) In [7] and [12], for a continuous piecewise linear function $f : \mathbb{R} \to \mathbb{R}$ to be tropical meromorphic, an additional requirement had been imposed upon that both one-sided derivatives of f were integers at each point $x \in \mathbb{R}$. In our paper [11], this additional requirement has been removed. Indeed, the authors are grateful to Prof. Aimo Hinkkanen for the idea of permitting real slopes in the definition of tropical meromorphic functions. A similar idea may be found in [7], p. 900, too.

(2) Observe that whenever $f : \mathbb{R} \to \mathbb{R}$ is a continuous piecewise linear function, then the discontinuities of f', see below, have no limit points in $\mathbb{R}$.

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A point x of derivative discontinuity of a tropical meromorphic function such that

$$\omega_f(x) := \lim_{\varepsilon \to 0+} \big(f'(x+\varepsilon) - f'(x-\varepsilon)\big) < 0$$

is said to be a *pole* of f of multiplicity $-\omega_f(x)$, while if $\omega_f(x) > 0$, then x is called a *root* (or a *zero-point*) of f of multiplicity $\omega_f(x)$. Observe that the multiplicity may be any nonnegative real number, to be denoted as $\tau_f(x)$ in what follows.

The basic notions of the Nevanlinna theory are now easily set up similarly as in [7] (See [11]):

The tropical proximity function for tropical meromorphic functions is defined as

$$m(r,f) := \frac{1}{2}(f^+(r) + f^+(-r)), \tag{1.1}$$

where $f^+(x) := \max\{f(x), 0\}$. Denoting by $n(r,f)$ the number of distinct poles of f in the interval $(-r,r)$, each pole multiplied by its multiplicity τ_f, the tropical counting function for the poles in $(-r,r)$ is defined as

$$N(r,f) := \frac{1}{2}\int_0^r n(t,f)dt = \frac{1}{2}\sum_{|b_\nu|<r} \tau_f(b_\nu)(r - |b_\nu|). \tag{1.2}$$

Defining then the tropical characteristic function $T(r,f)$ as usual,

$$T(r,f) := m(r,f) + N(r,f), \tag{1.3}$$

the tropical Poisson–Jensen formula, see [7], p. 5–6, to be proved below, readily implies the tropical Jensen formula

$$T(r,f) - T(r,-f) = f(0) \tag{1.4}$$

as a special case.

In this talk, we first recall basic results of Nevanlinna theory for tropical meromorphic functions, closely relying to what has been made in [7] by Halburd and Southall. As a novel element, not being included in [7], we propose a result that might be called the tropical second main theorem in [11].

2. Poisson–Jensen formula in the tropical setting

In what follows in this paper, a meromorphic function f is to be understood in the sense of Definition 1.1, unless otherwise specified. We may also call f to be *restricted* meromorphic, whenever all of its one-sided derivatives (slopes) are integers.

The Poisson–Jensen formula in the extended tropical setting is formally as in the restricted meromorphic case, see [7], Lemma 3.1. The same proof applies, but we however recall a complete proof here.

Theorem 2.1 ([7, 11])**.** *Suppose f is a meromorphic function on $[-r,r]$, for some $r > 0$ and denote the distinct roots, resp. poles, of f in this interval by a_μ, resp. by b_ν, with their corresponding multiplicities τ_f attached. Then for any $x \in (-r,r)$ we get the Poisson–Jensen formula*

$$\begin{aligned} f(x) &= \frac{f(r)+f(-r)}{2} + \frac{x}{2r}\{f(r) - f(-r)\} - \\ &\quad - \frac{1}{2r}\sum_{|a_\mu|<r} \tau_f(a_\mu)(r^2 - |a_\mu - x|r - a_\mu x) + \frac{1}{2r}\sum_{|b_\nu|<r} \tau_f(b_\nu)(r^2 - |b_\nu - x|r - b_\nu x), \end{aligned}$$

In the particular case of $x = 0$ we obtain the tropical Jensen formula

$$f(0) = \frac{f(r)+f(-r)}{2} - \frac{1}{2}\sum_{|a_\mu|<r} \tau_f(a_\mu)(r - |a_\mu|) + \frac{1}{2}\sum_{|b_\nu|<r} \tau_f(b_\nu)(r - |b_\nu|).$$

Proof. As in [7], we define an increasing sequence $(c_j), j = -p, \ldots, q$ in $(-r,r)$ in the following way. Let $c_0 = x$, and let the other points in this sequence be the points in $(-r,r)$ at which the derivative of f does not exist, i.e. f has either a root or a pole at these points. Further, we denote by m_j slopes of the line segments in the graph of f. In particular, we define $m_{j-1} := \lim_{x\to c_j^-} f'(x)$ for $j = -p, \dots, 0$, resp. $m_{j+1} := \lim_{x\to c_j^+} f'(x)$ for $j = 0, \dots, q$. Elementary geometric observation implies

$$\begin{aligned} f(r) - f(x) &= m_1(c_1 - x) + m_2(c_2 - c_1) + \cdots + m_q(c_q - c_{q-1}) + m_{q+1}(r - c_q) \\ &= -m_1 x + m_{q+1} r + c_1(m_1 - m_2) + \cdots + c_q(m_q - m_{q+1}) \\ &= m_1(r - x) - \sum_{j=1}^{q} (m_j - m_{j+1})(r - c_j). \end{aligned}$$

By a parallel reasoning,

$$f(x) - f(-r) = m_{-1}(r + x) - \sum_{j=1}^{p} (m_{-j-1} - m_{-j})(r + c_{-j}).$$

Multiplying the above two equalities by $(r + x)$ and $(r - x)$, respectively, and subtracting, we obtain

$$\begin{aligned} 2rf(x) &= r\big(f(r) + f(-r)\big) + x\big(f(r) - f(-r)\big) + (m_{-1} - m_1)(r^2 - x^2) \\ &\quad + \sum_{j=1}^{p} (m_{-j-1} - m_{-j})(r^2 - (x - c_{-j})r - c_{-j}x) + \\ &\quad + \sum_{j=1}^{q} (m_j - m_{j+1})(r^2 - (c_j - x)r - c_j x) \\ &= r\big(f(r) + f(-r)\big) + x\big(f(r) - f(-r)\big) + \sum_{c_j} -\omega_f(c_j)(r^2 - |c_j - x|r - c_j x). \end{aligned}$$

Recalling the definition of the multiplicity τ_f for roots and poles of f, the claim is an immediate consequence of this equality. □

3. Basic Nevanlinna theory in the tropical setting

It is verified in [11] that several basic inequalities, see [7], for the proximity function and the characteristic function hold in our present setting as well . In particular, the following simple observations are immediately proved by the corresponding definitions:

Lemma 3.1. *Let f and g be tropical meromorphic .*
(i) If $f \le g$, then $m(r, f) \le m(r, g)$.

(ii) Given a positive real number α, we see that

$$\clubsuit(r, f^{\otimes\alpha}) = \clubsuit(r, \alpha f) = \alpha\clubsuit(r, f)$$

holds for each $\clubsuit = m, N, T$ and for any value of r.

(iii) Each function $\clubsuit = m, N, T$ satisfies

$$\clubsuit(r, f \otimes g) \le \clubsuit(r, f) + \clubsuit(r, g),$$

for any value of r.

Remark. Observe that whenever $f \le g$, the inequality $N(r, f) \le N(r, g)$ is not necessarily true. Similarly, the inequality

$$N(r, f \oplus g) = N\big(r, \max(f, g)\big) \le \max\big(N(r, f), N(r, g)\big)$$

may fail. Indeed, as for the case $f \le g$, take f, g satisfying this inequality so that the graph of f is constant outside of $[-1, 1]$ and is $\wedge\!\wedge$-shaped in $[-1, 1]$, and let g be defined correspondingly as $\wedge$-shaped. Then f has two poles, while g has only one. If the slopes are suitably defined, then $N(r, f) > N(r, g)$. As for the case of $\max(f, g)$, a corresponding example is easily constructed. The corresponding observations are true for the characteristic function as well, provided just that the proximity functions are small enough.

As usual in the Nevanlinna theory, the next step from the Poisson–Jensen formula is to formulate the first main theorem. To this end, we recall the notation $L(f) := \inf\{f(b)\}$ over all poles b of f on $\mathbb{R}$, i.e.

$$L(f) := \inf\{f(b) : \omega_f(b) < 0, b \in \mathbb{R}\}.$$

In particular, if f has no poles (and so f is said to be tropical entire), then we have $L(f) = \inf\emptyset = +\infty$.

Theorem 3.2. *Let f be tropical meromorphic. Then*

$$\begin{aligned} T\big(r, 1_\circ \oslash (f \oplus a)\big) &= T\big(r, -\max(f,a)\big) \\ &\leq T(r,f) - N(r \mid f \geq a) + \max(a,0) - \max\big(f(0),a\big) \end{aligned}$$

for any $a \in \mathbb{R}$ and any value of r.
Moreover, an asymptotic equality

$$\begin{aligned} T\big(r, 1_\circ \oslash (f \oplus a)\big) &= T\big(r, -\max(f,a)\big) \\ &= T(r,f) - N(r \mid f \geq a) - \max\big(f(0),a\big) + \varepsilon(r,a) \end{aligned}$$

holds with $0 \leq \varepsilon(r,a) \leq \max(a,0)$ for each r. Here we denote

$$N(r \mid f \geq a) := N(r,f) - N\big(r, \max(f,a)\big)$$

which is always non-negative for each $a \in \mathbb{R}$ and vanishes identically when $-\infty < a < L(f)$ or when $L(f) = +\infty$.

Proof. Making use of the tropical Jensen formula (1.4), we immediately conclude

$$\begin{aligned} T\big(r, 1_\circ \oslash (f \oplus a)\big) &= T\big(r, -\max(f,a)\big) \\ &= T\big(r, \max(f,a)\big) - \max\big(f(0),a\big) \\ &\leq T(r,f) - N(r \mid f \geq a) + \max(a,0) - \max\big(f(0),a\big) \end{aligned}$$

for any $a \in \mathbb{R}$ and for any r. Here, together with

$$N\big(r, \max(f,a)\big) = N(r,f) - N(r \mid f \geq a),$$

we used the inequality $m(r, g \oplus h) = m\big(r, \max(g,h)\big) \leq m(r,g) + m(r,h)$ and the simple estimate $T(r,a) = \max(a,0)$. Note that $\max(a,0) - \max\big(f(0),a\big) \leq |a|$ holds.
Then we can obtain the asserted asymptotic equality in the following way:

$$\begin{aligned} T\big(r, 1_\circ \oslash (f \oplus a)\big) &= T\big(r, -\max(f,a)\big) \\ &= T\big(r, \max(f,a)\big) - \max\big(f(0),a\big) \\ &= m\big(, \max(f,a)\big) + N\big(r, \max(f,a)\big) - \max\big(f(0),a\big) \\ &\geq m(r,f) + N(r,f) - N(r \mid f \geq a) - \max\big(f(0),a\big) \\ &\geq T(r,f) - N(r \mid f \geq a) - \max\big(f(0),a\big), \end{aligned}$$

according to the monotonicity of $m(r, \bullet)$, Lemma 3.1, with respect to the second component $\bullet$. □

Example. As an example, for a non-constant linear function $f(x) = \alpha x + \beta$ with $\alpha > 0$ and $\beta > 0$, say, it immediately follows that

$$T(r,f) = m(r,f) = \begin{cases} \beta & \left(0 \leq r < \frac{\beta}{\alpha}\right) \\ \frac{\alpha}{2} r + \frac{\beta}{2} & \left(\frac{\beta}{\alpha} \leq r\right) \end{cases} .$$

It is a simple exercise to verify by this example that the error term $\varepsilon(r,a)$ in Theorem 3.2, may run over the whole interval $\big[0, \max\big(f(0),a\big)\big)$.

We next proceed to recall

Theorem 3.3 ([7, 11]). *The characteristic function $T(r,f)$ is a non-negative, continuous, non-decreasing piecewise linear function of r.*

Proof. The proof offered in [7], p. 894, applies verbatim. □

Remark. (1) The counting function $N(r,f)$ is a positive, continuous, non-decreasing piecewise linear function of r as well.

(2) In particular, Theorem 3.3 and Remark (1) above imply that standard Borel type theorems apply for $T(r,f)$ and $N(r,f)$, see e.g. [7], Lemma 3.5.

(3) As a remark for further needs, the following estimate, see [7], remains valid in the present setting as well: Indeed, for all $k > 1$,

$$n(r,f) \leq \frac{2}{(k-1)r} N(kr,f).$$

Moreover, given $\varepsilon > 0$, $R > 0$ and combining this estimate and a Borel type lemma, we get

$$n(r,f) \leq 4r^{-1}N(r,f)^{1+\varepsilon}$$

for all $r > R$ outside an exceptional set of finite logarithmic measure, see [7], Theorem 3.6.

(4) Defining a tropical rational function as a meromorphic function on $\mathbb{R}$ that has finitely many poles and roots only, the first estimate in (3) above may be used to show that a meromorphic function is rational if and only if $T(r,f) = O(r)$, see [7], Theorem 3.4.

Following the usual classical notion, a meromorphic function f is said to be of finite order of growth, if $T(r,f) \leq r^{\sigma}$ for some positive number σ, and for all r sufficiently large. Of course, this enables us to define the order $\rho(f)$ of a meromorphic function f in the usual way as

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

For example, we see easily the following

Proposition 3.4. *For any $c \in \mathbb{R}$, we have $N(r, x^{\otimes c}) = 0$ and*

$$T(r, x^{\otimes c}) = m(r, x^{\otimes c}) = \frac{|c|}{2} r \quad (r > 0) \quad \textit{thus} \quad \rho\big(x^{\otimes c}\big) = 1.$$

Proposition 3.5. *Any non-constant tropical periodic meromorphic function f satisfies $T(r,f) \asymp \kappa r^2$ for some $\kappa > 0$, hence $\rho(f) = 2$.*

In the finite order case, the characteristic function and the counting function of the shifts of meromorphic functions may be estimated by applying the following lemma, see [9], Lemma 3.2:

Lemma 3.6. *Let $T : [0,+\infty) \to [0,+\infty)$ be a non-decreasing continuous function of finite order ρ and take $c \in (0,+\infty)$. Then, given $\varepsilon > 0$, we have*

$$T(r+c) = T(r) + O\big(r^{\rho-1+\varepsilon}\big)$$

outside of a set of finite logarithmic measure.

Moreover, the estimates given in Remark (3) above, may be modified in the finite order situation as follows, see [7], Corollary 3.7:

Lemma 3.7 ([11])**.** *Let f be a meromorphic function of finite order, and suppose that $\delta < 1$ and $R > 0$. Then $n(r,f) \leq r^{-\delta}N(r,f)$ for all $r > R$ outside an exceptional set of finite logarithmic measure.*

In classical Nevanlinna theory and its applications, the lemma on logarithmic derivatives plays a fundamental role. It is likely that its tropical counterpart below, the lemma on tropical quotients of shifts, may become equally important:

Theorem 3.8 ([7, 11])**.** *Let f be tropical meromorphic. Then, for any $\varepsilon > 0$,*

$$m\big(r, f(x+c) \oslash f(x)\big) \leq \frac{2^{1+\varepsilon}14|c|}{r} \Big\{ T(r+|c|, f)^{1+\varepsilon} + o\big(T(r+|c|,f)\big) \Big\}$$

holds outside an exceptional set of finite logarithmic measure.

Proof. The proof given for Lemma 3.8 in [7], see p. 897–898, applies word by word. □

Another version of the lemma on tropical quotients of shifts is a tropical counterpart of a discussion in [6]:

Lemma 3.9 ([11])**.** *Let f be tropical meromorphic. Then for all $\alpha > 1$ and $r > 0$,*

$$m\big(r, f(x+c) \oslash f(x)\big) \leq \frac{20|c|/(\alpha-1)}{r+|c|} \left\{ T\big(\alpha(r+|c|), f\big) + |f(0)|/2 \right\}.$$

Proof. Following [7] as in the proof of their Lemma 3.8, by taking $\rho = (\alpha+1)(r+|c|)/2$ so that $\rho - r - |c| = (\alpha-1)(r+|c|)/2$ and $\rho > r + |c|$, we have

$$m\big(r, f(x+c) \oslash f(x)\big) \leq |c| \left\{ \frac{\big(m(\rho,f) + m(\rho,-f)\big)}{\rho} + \frac{3}{2}\big(n(\rho,f) + n(\rho,-f)\big) \right\}$$

for $x \in [-r, r]$. Since

$$\begin{aligned} N\big(\alpha(r+|c|), \pm f\big) &\geq \frac{1}{2}\int_{(\alpha+1)(r+|c|)/2}^{\alpha(r+|c|)} n(t, \pm f)dt \\ &\geq \frac{1}{2}n(\rho, \pm f)\frac{\alpha-1}{2}(r+|c|), \end{aligned}$$

we get

$$n(\rho, \pm f) \leq \frac{4}{\alpha-1}\frac{1}{r+|c|}N\big(\alpha(r+|c|), \pm f\big).$$

Since $\rho \leq \alpha(r+|c|)$ and $1 \leq 2(\alpha+1)/(\alpha-1)$, we may use the Jensen formula (1.4) to obtain the desired estimate by a simple computation. □

Corollary 3.10. *Let f be a meromorphic function of finite order ρ. Given $\varepsilon > 0$, f satisfies*

$$m\big(r, f(x+c) \oslash f(x)\big) = O(r^{\rho-1+\varepsilon})$$

outside an exceptional set of finite logarithmic measure.

4. Tropical meromorphic functions of hyper-order less than one

As pointed out in [6], a number of results in the difference variant of the Nevanlinna theory, see [4], typically expressed for meromorphic functions of finite order, may also be formulated for meromorphic functions of hyper-order less than one. As shown in [11], this extension applies to the tropical meromorphic setting as well. To this end, first recall the definition of hyper-order

$$\rho_2(f) := \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}. \tag{4.1}$$

Next recall the following lemma from [6], corresponding, in the case of hyper-order less than one, to our previous Lemma 3.4:

Lemma 4.1 ([6])**.** *Let $T : [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function and let $s \in (0, \infty)$. If the hyper-order of T is less than one, i.e.,*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \rho_2 < 1 \tag{4.2}$$

and $\delta \in (0, 1-\rho_2)$ then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^\delta}\right) \tag{4.3}$$

where r runs to infinity outside of a set of finite logarithmic measure.

For a proof of this lemma, see [6].

As a counterpart to Corollary 3.10, we may state the following

Proposition 4.2 ([11])**.** *Let f be a meromorphic function of hyper-order $\rho_2 < 1$. For any given $\delta \in (0, 1-\rho_2)$ and $c \in \mathbb{R}$, f satisfies*

$$m\big(r, f(x+c) \oslash f(x)\big) = o\big(T(r, f)/r^\delta\big),$$

as r approaches to infinity outside of a set of finite logarithmic measure.

Proof. In the finite order case, we have $\rho_2 = 0$, and so $\delta \in (0, 1)$. Take $\tau \in (\delta, 1)$. By [7], Theorem 3.10, we have

$$m\big(r, f(x+c) \oslash f(x)\big) = O\big(r^{-\tau}T(r, f)\big).$$

Therefore,

$$m\big(r, f(x+c) \oslash f(x)\big)\frac{r^\delta}{T(r, f)} = O\left(\frac{1}{r^{\tau-\delta}}\right) \to 0$$

as r approaches to infinity outside of a set of finite logarithmic measure. Therefore, we may assume that f is of infinite order. The proof for this case follows an idea from Halburd and Korhonen [4]. See also [6], p. 23–24. We first recall a generalized Borel Lemma as given in [1], Lemma 3.3.1: Let $\xi(x)$ and $\phi(s)$ be positive, nondecreasing and continuous functions defined for all sufficiently large x and s, respectively, and let $C > 1$. Then we have

$$T\left(s + \frac{\phi(s)}{\xi(T(s, f))}, f\right) \leq C\,T(s, f) \tag{4.4}$$

for all s outside of a set E satisfying

$$\int_{E\cap[s_0,R]}\frac{ds}{\phi(s)}\leq\frac{1}{\log C}\int_e^{T(R,f)}\frac{dx}{x\xi(x)}+O(1) \tag{4.5}$$

where $R<\infty$. Since f is of infinite order and of hyper-order less than 1, then by choosing $\phi(r)=r$, $\xi(x)=(\log x)^{1+\varepsilon}$ for $\varepsilon>0$ and

$$\alpha=1+\frac{\phi(r+|c|)}{(r+|c|)\xi(T(r+|c|,f))},$$

in Lemma 3.9, it follows that

$$m\big(r,f(x+c)\oslash f(z)\big)\leq\frac{12|c|\big(\log T(r+|c|,f)\big)^{1+\varepsilon}}{r+|c|}\left\{CT(r+|c|,f)+\frac{|f(0)|}{2}\right\} \tag{4.6}$$

as r approaches infinity outside of an r-set of finite logarithmic measure.
We may now fix $\varepsilon>0$ to satisfy $\delta=1-(\rho_2+\varepsilon)(1+\varepsilon)$. Then

$$\frac{\big(\log T(r+|c|,f)\big)^{1+\varepsilon}}{r+|c|}=o(1)(r+|c|)^{-\delta}$$

for all sufficiently large r. Then Lemma 4.1 and the above estimate (4.6) show that

$$m\big(r,f(x+c)\oslash f(z)\big)\leq\frac{13|c|}{r^\tau}T(r,f)=o(T(r,f)/r^\delta)$$

holds as r approaches infinity outside of a set of finite logarithmic measure. □

A *tropical hyper-exponential function* $e_\alpha(x)$ is found as a solution to equation

$$y(x+1)=y(x)^{\otimes\alpha},$$

see Section 6 below for its definition and basic properties. This function may be used to point out that the condition $\rho_2(f)<1$ cannot be dropped in general. In fact, we have for $\alpha>1$,

$$m\big(r,e_\alpha(z+1)\oslash e_\alpha(z)\big)=(\alpha-1)T(r,e_\alpha)$$

on the whole $\mathbb{R}$. Of course, Lemma 3.9 remains true for $f(x)=e_\alpha(x)$ as well.

5. Second main theorem in the tropical setting

In this section, we offer a tropical counterpart to the second main theorem. Observe, however, that the second main theorem in the tropical setting may not be as complete as in the usual Nevanlinna theory. This is due to the fact that certain elementary inequalities in the classical Nevanlinna theory, in particular those for the counting function, may fail in the tropical theory.

Theorem 5.1 ([11])**.** *Let f be a tropical meromorphic function and put $L(f):=\inf\{f(b):\omega_f(b)<0,\ b\in\mathbb{R}\}$. Given $c>0$, $q\in\mathbb{N}$ and q distinct values $a_j\in\mathbb{R}$ $(1\leq j\leq q)$ that satisfy $\max(a_1,\dots,a_q)<L(f)$, then*

$$\begin{aligned}qT(r,f)\leq\sum_{j=1}^q N\big(r,1_\circ\oslash(f\oplus a_j)\big)+T\big(r,f(x+c)\big)-N\big(r,1_\circ\oslash f(x+c)\big)+m\big(r,f(x+c)\oslash f(x)\big)-\\-f(c)+\sum_{j=1}^q\max\big(f(0),a_j\big)+p\max_{1\leq k\leq p}\max(a_k,0)+(p-1)\max_{1\leq k\leq p}\max(0,-a_k)\end{aligned} \tag{5.1}$$

holds for all $r>0$.

Before proceeding to prove Theorem 5.1, we define

$$N_1(r,f):=N\big(r,1_\circ\oslash f(x+c)\big)+2N(r,f)-N\big(r,f(x+c)\big). \tag{5.2}$$

Clearly, (5.2) is a tropical counterpart to the classical counting function

$$N_1(r,f):=N(r,1/f')+2N(r,f)-N(r,f')$$

for multiple values of f in the second main theorem for usual meromorphic functions. Using (5.2), we may write (5.1) as

$$qT(r,f) - T\big(r, f(x+c)\big) \leq \sum_{j=1}^{q} N\big(r, 1_\circ \oslash (f \oplus a_j)\big) - N_1(r,f) + 2N(r,f) - \\ - N\big(r, f(x+c)\big) + m\big(r, f(x+c) \oslash f(x)\big) + O(1). \quad (5.3)$$

Suppose now that f is of hyper-order $\rho_2 < 1$. Applying Lemma 4.1 to $T(r,f)$ and $N(r,f)$, and recalling Proposition 4.2 (with $\tau > 1 - \rho_2$), we obtain

Theorem 5.2 ([11])**.** *Suppose f is a nonconstant tropical meromorphic function of hyper-order $\rho_2 < 1$, and take $0 < \delta < 1 - \rho_2$. If $q(\geq 1)$ distinct values $a_1, \dots, a_q \in \mathbb{R}$ satisfy $\max(a_1, \dots, a_q) < L(f)$, then*

$$(q-1)T(r,f) \leq \sum_{j=1}^{q} N\big(r, 1_\circ \oslash (f \oplus a_j)\big) - N(r, 1_\circ \oslash f) + o\big(T(r,f)/r^\delta\big) \quad (5.4)$$

outside an exceptional set of finite logarithmic measure.

Proof. The desired inequality immediately follows from Theorem 5.1, combined with Proposition 4.2 and the next three inequalities, each of them being valid outside an exceptional set of finite logarithmic measure:

$$\begin{aligned} T\big(r, f(x+c)\big) &\leq T(r,f) + N\big(r, f(x+c)\big) - N(r,f) + o\big(T(r,f)/r^\delta\big), \\ N\big(r, f(x+c)\big) &\leq N(r+|c|, f) = N(r,f) + o\big(T(r,f)/r^\delta\big) \end{aligned}$$

and

$$N\big(r, 1_\circ \oslash f(x+c)\big) \geq N(r-|c|, 1_\circ \oslash f) = N(r, 1_\circ \oslash f) + o\big(T(r,f)/r^\delta\big).$$

These inequalities are immediate consequences of Lemma 4.1. □

Remark. Observe that whenever f is of finite order ρ, and so of hyper-order $\rho_2 = 0$, the error term $o\big(T(r,f)/r^\delta\big)$ in Theorem 5.2 may be replaced by $O(r^{\rho-1+\varepsilon})$ with $\varepsilon > 0$. Similarly in Corollary 5.3 below.

Corollary 5.3. *Suppose f is a nonconstant tropical meromorphic function of hyper-order $\rho_2 < 1$, and take $0 < \delta < 1 - \rho_2$. If $(q \geq 1)$ distinct values $a_1, \dots, a_q \in \mathbb{R}$ satisfy $\max(a_1, \dots, a_q) < L(f)$, and if $\ell(f) := \inf\{f(a) : \omega_f(a) > 0\} > -\infty$, then*

$$qT(r,f) \leq \sum_{j=1}^{q} N\big(r, 1_\circ \oslash (f \oplus a_j)\big) + o\big(T(r,f)/r^\delta\big) \quad (5.5)$$

outside an exceptional set of finite logarithmic measure. In particular,

$$T(r,f) \leq N\big(r, 1_\circ \oslash (f \oplus a)\big) + o\big(T(r,f)/r^\delta\big) \quad (5.6)$$

holds for all $a \in \mathbb{R}$ such that $a < L(f)$.

Proof. Let $a_1, \dots, a_q$ be q distinct real values such that $a_j < L(f)$, $1 \leq j \leq q$. In order to prove the assertion (5.5), choose a real number μ such that

$$\mu < \min\Big\{\max(a_1, \dots, a_q), \ell(f)\Big\}$$

and put

$$g(x) := f(x) - \mu, \quad \tilde{a}_j := a_j - \mu (> 0)\ (1 \leq j \leq q) \quad \text{and} \quad \tilde{a}_0 := 0.$$

Then the following observations are easily checked:

- $\rho_2(g) = \rho_2(f)$ as well as $\rho(g) = \rho(f)$,
- $\omega_g(x) \equiv \omega_f(x)$,
- $L(g) := \inf\{g(b) : \omega_g(b) < 0\} = L(f) - \mu$,
- $L(g) - \max(\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_q) = L(f) - \max(a_1, \dots, a_q) > 0$,
- $\ell(g) := \inf\{g(a) : \omega_g(a) > 0\} = \ell(f) - \mu > 0$.

We now apply Theorem 5.2 to the function $g(x)$ and the $q+1$ distinct values $\tilde{a}_j$ to obtain

$$qT(r,g) \le \sum_{j=0}^{q} N\big(r, 1_\circ \oslash (g \oplus \tilde{a}_j)\big) - N(r, 1_\circ \oslash g) + S_*(r,g) \tag{5.7}$$

outside an exceptional set of finite logarithmic measure. Since $\ell(g) > 0$, the two functions $1_\circ \oslash (g \oplus 0) = -\max(g, 0)$ and $1_\circ \oslash g = -g$ have exactly the same poles, and therefore

$$N\big(r, 1_\circ \oslash (g \oplus \tilde{a}_0)\big) - N(r, 1_\circ \oslash g) \equiv 0$$

in the above inequality. Since $T(r, f-\mu) \ge T(r,f) - \mu$ and $g \oplus \tilde{a}_j = (f \oplus a_j) - \mu$, we obtain the desired estimate (5.5) for the original function f. □

Remark. We should perhaps point out here that by this corollary, nonconstant tropical meromorphic functions of hyper-order $\rho_2 < 1$ and satisfying $\ell(f) \ne -\infty$ have no deficient values $a < L(f)$ in the sense that

$$1 - \limsup_{r\to\infty} \frac{N(r, 1_\circ \oslash (f \oplus a))}{T(r,f)} = 0.$$

However, omitted values may well appear. For example, any linear function omits both roots and poles. Moreover, rational functions of shape $\wedge$, resp. of $\vee$, also omit roots, resp. poles. Indeed, the estimates (5.1), (5.5) and (5.6) above do not include, in general, consideration of the roots, resp. the poles. In fact, the counting functions $N(r, 1_\circ \oslash f)$ and $N\big(r, 1_\circ \oslash (f \oplus 0)\big)$ do not coincide in general, since $1_\circ \oslash f = 1_\circ \oslash (f \oplus (-\infty))$. Therefore, $1_\circ \oslash f$ and $1_\circ \oslash (f \oplus 0)$ do not coincide in general. Note also that $m\big(r, 1_\circ \oslash (f \oplus 0)\big) = 0$, since $1_\circ \oslash (f \oplus 0) = -f^+ \le 0$, hence $N\big(r, 1_\circ \oslash (f \oplus 0)\big) = T\big(r, 1_\circ \oslash (f \oplus 0)\big) = T(r,f) - \max\big(f(0), 0\big)$ by the first main theorem, Theorem 3.2, for any tropical meromorphic function f. On the other hand, to get $N(r, 1_\circ \oslash f) = T(r,f) - \max\big(f(0), 0\big)$, we need to have $\ell(f) \ge 0$.

To illustrate these comments, consider the linear function $f(x) = x+1$, taking r large enough, and $q = 1$, $a_1 = 0$ and $c > 0$. Then we have $L(f) = \ell(f) = +\infty$ and

$$N(r,f) = N(r, 1_\circ \oslash f) = N\big(r, f(x+c)\big) = N\big(r, 1_\circ \oslash f(x+c)\big) \equiv 0.$$

Moreover, $T(r,f) = m(r,f) = \frac{r+1}{2}$, $T\big(r, f(x+c)\big) = m\big(r, f(x+c)\big) = \frac{r+c+1}{2}$ and $m\big(r, f(x+c) \oslash f(x)\big) = c$. Therefore, (5.1) takes the form

$$\frac{r+1}{2} \le r + \frac{c+1}{2},$$

while (5.5) in Corollary 5.3 becomes

$$\frac{r+1}{2} \le \frac{r}{2} + O(r^\varepsilon)$$

for any $\varepsilon > 0$.

Finally, we remark that the assumption $\rho_2 < 1$ above cannot be deleted. To see this, the reader may consider the above-mentioned tropical hyper-exponential functions e_α. In particular, such a function may have uncountably many deficient values.

Remark. The estimate (5.1) given in Theorem 5.1 may indeed be written in the form

$$\sum_{k=1}^{q} m\Big(r, 1_\circ \oslash (f \oplus a_k)\Big) \le m\big(r, 1_\circ \oslash f(x+c)\big) + m\left(r, \bigoplus_{k=1}^{q} f(x+c) \oslash \big(f(x) \oplus a_k\big)\right) + O(1)\,. \tag{5.8}$$

This is an obvious tropical counterpart to the classical inequality

$$\sum_{k=1}^{q} m\Big(r, 1/\big(f(z) - a_k\big)\Big) \le m\big(r, 1/f'(z)\big) + m\left(r, \sum_{k=1}^{q} \frac{f'(z)}{f(z) - a_k}\right) + O(1)\,,$$

see e.g. [8], 32–33. On the other hand, Theorem 5.2 and Corollary 5.3 are reminiscent of the classical second main theorem.

In order to prove Theorem 5.1, we prepare a sequence of lemmas.

Lemma 5.4. *For any $p \in \mathbb{N}$, any $a_k \in \mathbb{R}$ $(1 \le k \le p)$ and any $c \in \mathbb{R} \setminus \{0\}$, we have*

$$m\left(r, 1_\circ \oslash \big(\bigotimes_{k=1}^{p}(f \oplus a_k)\big)\right) \le T\big(r, f(x+c)\big) - N\big(r, 1_\circ \oslash f(x+c)\big) + \\ + m\left(r, f(x+c) \oslash \Big(\bigotimes_{k=1}^{p}\big(f(x) \oplus a_k\big)\Big)\right) - f(c)\,.$$

Proof. Note that

$$1_\circ \oslash \big(\bigotimes_{k=1}^{p}(f \oplus a_k)\big) = \big(1_\circ \oslash f(x+c)\big) \otimes \left(f(x+c) \oslash \big(\bigotimes_{k=1}^{p}(f \oplus a_k)\big)\right).$$

Since $m(r, g \otimes h) \le m(r, g) + m(r, h)$, we have

$$\begin{aligned} & m\left(r, 1_\circ \oslash \big(\bigotimes_{k=1}^{p}(f \oplus a_k)\big)\right) \\ \le\ & m\big(r, 1_\circ \oslash f(x+c)\big) + m\left(r, f(x+c) \oslash \big(\bigotimes_{k=1}^{p}(f \oplus a_k)\big)\right) \\ =\ & T\big(r, f(x+c)\big) - f(0+c) - N\big(r, 1_\circ \oslash f(x+c)\big) + m\left(r, f(x+c) \oslash \big(\bigotimes_{k=1}^{p}(f \oplus a_k)\big)\right), \end{aligned}$$

by using Jensen's formula. □

The following inequality is important below as a replacement to the usual partial fraction decomposition applied in the proof of the classical Second Main Theorem:

Lemma 5.5. *For any $p \in \mathbb{N}$, any $a_k \in \mathbb{R}$ $(1 \le k \le p)$, and for $b_k := 1_\circ \oslash \big(\bigotimes_{j \ne k}(a_j \oplus a_k)\big)$ $(1 \le k \le p)$, we have*

$$1_\circ \oslash \big(\bigotimes_{k=1}^{p}(x \oplus a_k)\big) \le \bigoplus_{k=1}^{p}\big(b_k \oslash (x \oplus a_k)\big)\,, \tag{5.9}$$

for any $x \in \mathbb{R}$.

Proof. Inequality (5.9) is equivalent to

$$\sum_{k=1}^{p} \max(x, a_k) \ge \min_{1 \le k \le p}\left(\max(x, a_k) + \sum_{j \ne k} \max(a_j, a_k)\right)\,. \tag{5.10}$$

Here, we may assume without loss of generality

$$a_1 \le a_2 \le \cdots \le a_p.$$

Case i): Suppose first that $x \le a_1$. Then

$$\max(x, a_k) = a_k \quad (1 \le k \le p)$$

and thus the left-hand side of (5.10) becomes $\sum_{k=1}^{p} a_k$, while the right-hand side of (5.10) becomes

$$\min_{1 \le k \le p}\left(a_k + (k-1)a_k + \sum_{j=k+1}^{p} a_j\right) = a_1 + \sum_{j=2}^{p} a_j = \sum_{k=1}^{p} a_k\,,$$

also. Hence (5.10) holds in this case.

Case ii): If $a_{\ell-1} \le x \le a_\ell$ for some $2 \le \ell \le p$, then

$$\max(x, a_k) = \begin{cases} x & (1 \le k \le \ell - 1) \\ a_k & (\ell \le k \le p) \end{cases}\,.$$

Thus the left-hand side of (5.10) becomes

$$(\ell - 1)x + \sum_{k=\ell}^{p} a_k \ge \sum_{1 \le j \le \ell-1} a_j + \sum_{k=\ell}^{p} a_k = \sum_{j=1}^{p} a_j\,,$$

which is not less than the right-hand side of (5.10). In fact, the latter is the minimum of the set

$$\left\{ x + \sum_{j=2}^{p} a_j\,,\ \ldots\,,\ x + (\ell-2)a_{\ell-1} + \sum_{\ell \le j \le p} a_j\,,\ a_\ell + (\ell-1)a_\ell + \sum_{\ell+1 \le j \le p} a_j\,,\ \ldots\ ,\ \sum_{j=1}^{p} a_j \right\},$$

which verifies (5.10) in this case.

Case iii): If $a_p \le x$, then we have

$$\max(x, a_k) = x \quad (1 \le k \le p)\,.$$

The left-hand side of (5.10) is now px, while the right-hand side of (5.10) becomes

$$\min_{1 \le k \le p} \left\{ x + (k-1)a_k + \sum_{j=k+1}^{p} a_j \right\} \le x + (k-1)x + (p-k)x = px\,,$$

proving the remaining case of (5.10). □

Lemma 5.6. *For any $p \in \mathbb{N}$ and any $a_k \in \mathbb{R}$ $(1 \le k \le p)$ and any $c \in \mathbb{R} \setminus \{0\}$, we have*

$$\begin{aligned} m\Big(r, f(x+c) \oslash \big(\bigotimes_{k=1}^{p} (f(x) \oplus a_k)\big)\Big) \\ \le m\left(r, \bigoplus_{k=1}^{p} \Big(f(x+c) \oslash \big(f(x) \oplus a_k\big)\Big)\right) + (p-1) \max_{1 \le k \le p} \big(1_\circ \oplus (1_\circ \oslash a_k)\big). \end{aligned}$$

Proof. First, applying Lemma 5.5, we see

$$\begin{aligned} f(x+c) \oslash \left(\bigotimes_{k=1}^{p} \big(f(x) \oplus a_k\big)\right) &\le f(x+c) \otimes \Big(\bigoplus_{k=1}^{p} b_k \oslash \big(f(x) \oplus a_k\big)\Big) \\ &= \bigoplus_{k=1}^{p} \Big\{ f(x+c) \otimes \Big(b_k \oslash \big(f(x) \oplus a_k\big)\Big)\Big\} \\ &= \max_{1 \le k \le p} \Big\{ f(x+c) + \Big(b_k - \max\big(f(x), a_k\big)\Big)\Big\} \\ &= \max_{1 \le k \le p} \Big\{ b_k + \Big(f(x+c) - \max\big(f(x), a_k\big)\Big)\Big\} \\ &\le \max_{1 \le k \le p} \Big\{ 1_\circ \oslash \big(\bigotimes_{j \ne k} (a_j \oplus a_k)\big)\Big\} + \bigoplus_{k=1}^{p} \Big(f(x+c) \oslash \big(f(x) \oplus a_k\big)\Big)\,. \end{aligned}$$

Note

$$\begin{aligned} \left(\max_{1 \le k \le p} \Big\{ 1_\circ \oslash \big(\bigotimes_{j \ne k} (a_j \oplus a_k)\big)\Big\}\right)^+ &= \left(\max_{1 \le k \le p} \Big\{ \sum_{j \ne k} \min(-a_j, -a_k)\Big\}\right)^+ \\ &\le \Big\{(p-1) \max_{1 \le k \le p} (-a_k)\Big\}^+ \le (p-1) \max_{1 \le k \le p} (-a_k)^+ \end{aligned}$$

By monotonicity of $m(r, *)$, the asserted inequality follows from the definition of the proximity function. □

Remark. Observe that $f(x+c) \oslash \big(f(x) \oplus a\big) \le f(x+c) \oslash f(x)$ for each $a \in \mathbb{R}$.

Lemma 5.7. *For any $p \in \mathbb{N}$ and any $a_k \in \mathbb{R}$ $(1 \le k \le p)$, we have*

$$\begin{aligned} T\Big(r, 1_\circ \oslash \big(\bigotimes_{k=1}^{p} (f \oplus a_k)\big)\Big) &= m\Big(r, 1_\circ \oslash \big(\otimes_{k=1}^{p} (f \oplus a_k)\big)\Big) + N\Big(r, 1_\circ \oslash \big(\bigotimes_{k=1}^{p} (f \oplus a_k)\big)\Big) \\ &= T\Big(r, \bigotimes_{k=1}^{p} (f \oplus a_k)\Big) - \bigotimes_{k=1}^{p} \big(f(0) \oplus a_k\big)\,. \end{aligned}$$

Proof. We may apply the tropical Jensen formula to the function $F(x) := \bigotimes_{k=1}^{p}(f \oplus a_k)$ to obtain the above identity. Note that

$$F(0) = \bigotimes_{k=1}^{p}\big(f(0) \oplus a_k\big) = \sum_{k=1}^{p} \max\big(f(0), a_k\big).$$

□

Lemma 5.8. *For any $p \in \mathbb{N}$ and any $a_k \in \mathbb{R}$ $(1 \le k \le p)$, we have*

$$N\Bigg(r, 1_\circ \oslash \big(\bigotimes_{k=1}^{p}(f \oplus a_k)\big)\Bigg) \le \sum_{k=1}^{p} N\big(r, 1_\circ \oslash (f \oplus a_k)\big)\,.$$

Proof. First, recall the linearity of $\omega_f(x)$ at each point x with respect to f, that is,

$$\omega_{g+h}(x) = \omega_g(x) + \omega_h(x)\,.$$

Therefore,

$$\max\{\omega_{g+h}(x), 0\} \le \max\{\omega_g(x), 0\} + \max\{\omega_h(x), 0\}$$

for any $x \in \mathbb{R}$, and so $n(t, g+h) \le n(t, g) + n(t, h)$ for any r. Hence,

$$N(r, g \otimes h) \le N(r, g) + N(r, h)$$

holds for any r. The desired inequality now follows from

$$1_\circ \oslash \big(\bigotimes_{k=1}^{p}(f \oplus a_k)\big) = -\sum_{k=1}^{p}(f \oplus a_k) = \sum_{k=1}^{p}\big\{1_\circ \oslash (f \oplus a_k)\big\} = \bigotimes_{k=1}^{p}\big\{1_\circ \oslash (f \oplus a_k)\big\}\,.$$

□

Lemma 5.9. *For any $p \in \mathbb{N}$ and any $a_k \in \mathbb{R}$ $(1 \le k \le p)$, we have*

$$T\big(r, \bigotimes_{k=1}^{p}(f \oplus a_k)\big) \le p\,T(r, f) + \sum_{k=1}^{p} \max(a_k, 0)\,.$$

Proof. A straightforward reasoning by using $T(r, g \otimes h) \le T(r, g) + T(r, h)$ and $T(r, g \oplus h) \le T(r, g) + T(r, h)$ directly confirms that

$$\begin{aligned} T\big(r, \bigotimes_{k=1}^{p}(f \oplus a_k)\big) &\le \sum_{k=1}^{p} T(r, f \oplus a_k) \le \sum_{k=1}^{p}\big\{T(r, f) + T(r, a_k)\big\} \\ &= p\,T(r, f) + \sum_{k=1}^{p} \max(a_k, 0)\,, \end{aligned}$$

since $T(r, a) = m(r, a) = \max(a, 0)$ for any constant $a \in \mathbb{R}$. □

In order to show a related reversed inequality to Lemma 5.9, we first prove the following

Lemma 5.10. *For any $p \in \mathbb{N}$ and any $a_k \in \mathbb{R}$ $(1 \le k \le p)$, we have*

$$\max\left\{\sum_{k=1}^{p} \max(f, a_k)), p\, \max(a_1, \dots, a_p)\right\} = p\, \max\big(f, \max(a_1, \dots, a_p)\big)\,,$$

that is,

$$\Big(\bigotimes_{k=1}^{p}(f \oplus a_k)\Big) \oplus \Big(\bigoplus_{k=1}^{p} a_k\Big)^{\otimes p} = \Big(f \oplus \big(\bigoplus_{k=1}^{p} a_k\big)\Big)^{\otimes p}\,.$$

Proof. If $f \le \max(a_1, \dots, a_p)$, then

$$\sum_{k=1}^{p} \max(f, a_k) \le p\, \max(a_1, \dots, a_p),$$

while if $f > \max(a_1, \dots, a_p)$, then

$$\sum_{k=1}^{p} \max(f, a_k) \ge p\, \max(a_1, \dots, a_p)$$

The assertion immediately follows. □

As an application of Lemma 5.10, we obtain

Lemma 5.11. *For any $p \in \mathbb{N}$ and any $a_k \in \mathbb{R}$ $(1 \le k \le p)$ with all $a_k < L_{\mathbb{I}}(f)$, we have*

$$T\big(r, \bigotimes_{k=1}^{p}(f \oplus a_k)\big) \ge p\,T(r,f) - p \max_{1\le k\le p} \max(a_k, 0)\,.$$

Proof. By Lemma 5.10 together with $T(r, g^{\otimes p}) = p\,T(r,g)$, we have

$$T\left(r, (\bigotimes_{k=1}^{p}(f \oplus a_k)) \oplus (\bigoplus_{k=1}^{p} a_k)^{\otimes p}\right) = p\,T\left(r, f \oplus (\bigoplus_{k=1}^{p} a_k)\right).$$

Clearly,

$$p\,T\left(r, f \oplus (\bigoplus_{k=1}^{p} a_k)\right) \ge p\,T(r,f).$$

In fact, since $f \oplus \big(\bigoplus_{k=1}^{p} a_k\big) = \max\big\{f, \bigoplus_{k=1}^{p} a_k\big\} \ge f$, we see that

$$m\left(r, f \oplus (\bigoplus_{k=1}^{p} a_k)\right) \ge m(r,f)\,,$$

while

$$N\left(r, f \oplus (\bigoplus_{k=1}^{p} a_k)\right) = N(r,f)$$

holds, since $\bigoplus_{k=1}^{p} a_k = \max(a_1, \dots, a_p) < L(f)$.

On the other hand, since $T(r, g \oplus h) \le T(r,g) + T(r,h)$, we have

$$T\left(r, (\bigotimes_{k=1}^{p}(f \oplus a_k)) \oplus (\bigoplus_{k=1}^{p} a_k)^{\otimes p}\right) \le T\left(r, \bigotimes_{k=1}^{p}(f \oplus a_k)\right) + p\,T\left(r, \bigoplus_{k=1}^{p} a_k\right).$$

Recalling again

$$T\left(r, \bigoplus_{k=1}^{p} a_k\right) = \max\big((\max_{k=1}^{p} a_k), 0)\big) = \max_{1\le k\le p} \max(a_k, 0),$$

we obtain

$$T\left(r, \bigotimes_{k=1}^{p}(f \oplus a_k)\right) + p \max_{1\le k\le p} \max(a_k, 0) \ge p\,T(r,f)\,,$$

as desired. $\square$

Remark. We now have the estimate

$$\left| T\big(r, \bigotimes_{k=1}^{p}(f \oplus a_k)\big) - p\,T(r,f) \right| \le p \max_{k=1}^{p} \max(a_k, 0)\,.$$

under the assumption $\max(a_1, \dots, a_p) < L(f)$. Therefore, $T\big(r, \bigotimes_{k=1}^{p}(f \oplus a_k)\big) = p\,T(r,f)$ whenever $a_k \le 0$ for each $k = 1, \dots, p$. This may seem a bit curious. Recall, however, the assumption $a_k < L(f)$ and its strong consequence $N(r, f \oplus a_k) = N(r,f)$ for each k.

Proof of Theorem 5.1. It follows by combining Lemmas 5.4, 5.6 and 5.7 - 5.11 above that

$$\begin{aligned} & T\big(r, f(x+c)\big) - N\big(r, 1_\circ \oslash f(x+c)\big) + m\big(r, f(x+c) \oslash f(x)\big) - f(c) \\ \ge\ & pT(r,f) - \sum_{k=1}^{p} N\big(r, 1_\circ \oslash (f \oplus a_k)\big) - \sum_{k=1}^{p} \max\big(f(0), a_k\big) - p \max_{1\le k\le p} \max(a_k, 0) - (p-1) \max_{1\le k\le p} \max(0, -a_k)\,, \end{aligned}$$

and therefore

$$\begin{aligned} pT(r,f) \le & \sum_{k=1}^{p} N\big(r, 1_\circ \oslash (f \oplus a_k)\big) + T\big(r, f(x+c)\big) - N\big(r, 1_\circ \oslash f(x+c)\big) + m\big(r, f(x+c) \oslash f(x)\big) \\ & - f(c) + \sum_{k=1}^{p} \max\big(f(0), a_k\big) + p \max_{1\le k\le p} \max(a_k, 0) + (p-1) \max_{1\le k\le p} \max(0, -a_k)\,, \end{aligned} \tag{5.1}$$

completing the proof.

6. Tropical hyper-exponential functions

We consider certain special tropical meromorphic functions, which are reminiscent to hyper-exponential functions $\exp(z^c)$ over the usual algebra.

Definition 6.1. Let α be a real number with $|\alpha| > 1$. Define a function $e_\alpha(x)$ on $\mathbb{R}$ by

$$e_\alpha(x) := \alpha^{[x]}(x-[x]) + \sum_{j=-\infty}^{[x]-1} \alpha^j = \alpha^{[x]}\left(x - [x] + \frac{1}{\alpha - 1}\right).$$

Then we see

Proposition 6.2. *The function $e_\alpha(x)$ is tropical meromorphic on $\mathbb{R}$ satisfying*

- $e_\alpha(m) = \alpha^m/(\alpha-1)$ *for each $m \in \mathbb{Z}$,*
- $e_\alpha(x) = x + \frac{1}{\alpha-1}$ *for any $x \in [0,1)$, and*
- *the functional equation $y(x+1) = y(x)^{\otimes\alpha}$ on the whole $\mathbb{R}$.*

In a similar way, for a real number with $|\beta| < 1$, we consider a function

$$e_\beta(x) \;:=\; \sum_{j=[x]}^{\infty} \beta^j - \beta^{[x]}(x-[x]) = \sum_{j=[x]+1}^{\infty} \beta^j + \beta^{[x]}(1-x+[x]) = \beta^{[x]}\left(\frac{1}{1-\beta} - x + [x]\right).$$

Then we similarly obtain

Proposition 6.3. *This function $e_\beta(x)$ is also tropical meromorphic on $\mathbb{R}$ and satisfies*

- $e_\beta(m) = \beta^m/(1-\beta)$ *for each $m \in \mathbb{Z}$,*
- $e_\beta(x) = -x + \frac{1}{1-\beta}$ *for any $x \in [0,1)$, and*
- *the functional equation $y(x+1) = y(x)^{\otimes\beta}$ on the whole $\mathbb{R}$.*

As for the connection between $e_\alpha(x)$ with $|\alpha| > 1$ and $e_\beta(x)$ with $|\beta| < 1$, we obtain

Proposition 6.4. *Suppose $\alpha \neq \pm 1$. Then*

- $e_\alpha(-x) = \frac{1}{\alpha} e_{1/\alpha}(x)$, *and*
- $e_\alpha(0) = \frac{1}{\alpha} e_{1/\alpha}(0)$.

Remark. The slopes of $e_\alpha(x)$ range over the infinite set $\{\alpha^j \mid -\infty < j < +\infty\}$ for all $\alpha \neq \pm 1$.

Proposition 6.5. *The function $e_\alpha(x)$, $\alpha \neq \pm 1$, is of infinite order and, in fact, of hyper-order one.*

Remark. Observe that if $\alpha > 1$, then $e_\alpha(x) \oplus a > 0$ for any $a \in \mathbb{R}$. This shows that $m\big(r, 1_\circ \oslash (e_\alpha \oplus a)\big) = m\big(r, -(e_\alpha \oplus a)\big) \equiv 0$, so that $T\big(r, 1_\circ \oslash (e_\alpha \oplus a)\big) = N\big(r, 1_\circ \oslash (e_\alpha \oplus a)\big)$. On the other hand, if $\alpha < -1$, then $e_\alpha(x)$ has a zero of multiplicity $\alpha^{2j}(1-1/\alpha)$ at each even integer $x = 2j$ and a pole of multiplicity $\alpha^{2j}(1-\alpha)$ at each odd integer $x = 2j+1$, since $\omega_{e_\alpha}(m) = \alpha^m(1-1/\alpha)$ for each $m \in \mathbb{Z}$. Thus we see that when $2\ell \le t < 2(\ell+1)$ for some integer ℓ, that is, when $\ell = \left[\frac{t}{2}\right]$, then

$$\begin{aligned} n(t, 1_\circ \oslash e_\alpha) &= \sum_{j=-\ell}^{\ell} \alpha^{2j}\left(1 - \frac{1}{\alpha}\right) = \left(1 - \frac{1}{\alpha}\right)\left\{\frac{\alpha^2}{\alpha^2-1}\alpha^{2\ell} + \frac{1}{1-\alpha^2}\alpha^{-2\ell}\right\} \\ &= \frac{\alpha}{\alpha+1}\alpha^{2[t/2]} - \frac{1}{\alpha(\alpha+1)}\alpha^{-2[t/2]} \ge \frac{1}{\alpha(\alpha+1)}|\alpha|^t - \frac{\alpha}{\alpha+1}|\alpha|^{-t} \end{aligned}$$

so that $N(r, e_\alpha) \ge |\alpha|^r/\{2\alpha(\alpha+1)\log\alpha\} + O(1)$.

As for the case of $e_\beta(x)$, its slopes again range over the infinite set $\{\beta^j \mid -\infty < j < +\infty\}$. When $0 < \beta < 1$, this function has no poles and $m(r, e_\beta) \asymp C\beta^{-r}$. In the case of $-1 < \beta < 0$, it has a pole of multiplicity $\beta^{2j}(1-1/\beta)$ at each even integer $x = 2j$ and a root of multiplicity $\beta^{2j}(1-\beta)$ at each odd integer $x = 2j+1$. In particular, taking $\beta = -1/2$, it is immediate to obtain that

$$N\big(r, e_\beta(x)\big) = 2N\big(r, 1_\circ \oslash e_\beta(x)\big) + O(r).$$

In order to see that the assertion of Corollary 5.3 fails for $e_\beta(x)$ with $\beta = -1/2$, take $a = -1 < 0 = L(e_\beta)$. Then the roots of $e_\beta(x) \oplus a$ are the same as those of $e_\beta(x)$ for all $x = 2j+1 > 0$, while for $x = 2j+1 < 0$, each such root of $e_\beta(x)$, having multiplicity $\beta^{2j}(1-\beta)$, splits into two roots of $e_\beta(x) \oplus a$, with the sum of their multiplicities being equal to $\beta^{2j}(1-\beta)$. Therefore, we have

$$T\big(r, e_\beta(x)\big) \ge N\big(r, e_\beta(x)\big) = 2N\Big(r, 1_\circ \oslash \big(e_\beta(x) \oplus a\big)\Big) + O(r). \tag{6.1}$$

More generally, the same conclusion as in (6.1) follows for all $a < 0$. In particular, this means that each $a < 0$ is a deficient value for $e_\beta(x)$ in the sense that

$$1 - \limsup_{r\to\infty} \frac{N\Big(r, 1_\circ \oslash \big(e_\beta(x) \oplus a\big)\Big)}{T\big(r, e_\beta(x)\big)} \geq \frac{1}{2} > 0.$$

7. Concluding remarks

In [7] and [11], they give an application to determine the possible tropical meromorphic solutions to some ultra-discrete equations.

In [12] and [11], they observe tropical counterparts of three key lemmas from Nevanlinna theory, frequently applied to complex differential and difference equations, namely the Valiron–Mohon'ko lemma, the Mohon'ko lemma and the Clunie lemma, see e.g., respectively, [13], p. 83, [3], Lemma 2, and [14], Theorem 6.

Possible counterparts to Cartan theory for holomorphic curves to a projective space and Nevanlinna theory in an annulus are also under investigation.

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