Loewner Equation and SLE: Overview

Applications of Loewner Equation

Loewner 方程式の応用について
Applications of Loewner Equation

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Let $\mathcal{O}_0$ be the set of analytic germs $f(z)$ of one complex variable $z$ with $f(0) = 0$ and let

$$\mathcal{O}_0^\times = \{ f \in \mathcal{O}_0 : f'(0) \neq 0 \}.$$
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The set $\mathcal{O}_0$ has a structure of semigroup concerning the composition and $\mathcal{O}_0^\times$ becomes a subgroup. Our main aim is to interpret the Loewner equation in the context of these structures.

Let \( f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots \) be an element of \( \mathcal{O}_0^\times \). The Grunsky coefficients \( A_{m,n} \) of \( f \) are defined by

\[
\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=0}^{\infty} A_{m,n} z^m \zeta^n
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for small enough \( z, \zeta \).
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We also consider the expansion

$$\log \frac{1}{1 - f(z) f(\zeta)} = \sum_{m,n=0}^{\infty} B_{m,n} z^m \zeta^n.$$
Grunsky-Nehari inequality

Theorem (Nehari 1953)

If $f : \mathbb{D} \to \mathbb{D}$ is univalent and $f(0) = 0$, then

$$\text{Re} \sum_{l,m=1}^{N} (A_{l,m} t_l t_m + B_{l,m} t_l \bar{t}_m) \leq \sum_{l=1}^{N} \frac{|t_l|^2}{l}$$

for $t_1, \ldots, t_N \in \mathbb{C}$. 

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Schiffer and Tammi extended it by using the power matrix and Loewner equation.
For $f \in \mathcal{O}_0$, we consider the expansion

$$f(z)^m = \sum_{n=1}^{\infty} [f]^m_n z^n$$

for a natural number $m$. 

Note that $[f]^m_n = 0$ for $n < m$. 

Consider the matrix $[f]$ with entries $[f]^m_n (m, n = 1, 2, \ldots)$. This is called the power matrix of $f$. 

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Let
\[ f_t(z) = \sum_{k=1}^{\infty} c_k(t) z^k \]
be a solution to the (radial) Loewner equation
\[ \dot{f}_t = -z \frac{1 + u(t)z}{1 - u(t)z} f'_t, \quad t \geq 0, \]
for a continuous \( u(t) \) with \( |u(t)| = 1 \). Then,
\[ \sum_{k=1}^{\infty} \dot{c}_k(t) z^k = -(1 + 2 \sum_{k=1}^{\infty} u^k z^k) \sum_{k=1}^{\infty} k c_k(t) z^k. \]
Therefore,

\[ \dot{c}_k(t) = -kc_k(t) - 2 \sum_{m=1}^{k-1} mu(t)^{k-m} c_m(t) \]

for \( k \geq 1. \) (In particular, \( c_1(t) = e^{-t}. \))
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for \( k \geq n \), where \( c_{n,k}(t) = [f_t]^n_k \). From these equations, Schiffer and Tammi (1971) obtained an extension of Grunsky-Nehari inequalities.
The power matrix has the following remarkable property:

\[ [f \circ g] = [f][g] \]

for \( f, g \in \mathcal{O}_0 \). Also, \([\text{id}] = \text{id}\). Therefore, this gives a matrix representation of \( \mathcal{O}_0 \) and \( \mathcal{O}_0^\times \).
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Also note that

\[n[f]^{k+1} = (k + 1) \sum_{m=1}^{n-k} m[f]_m^1 [f]_n^k [f]_{n-m}^k.\]
Lie group $G$ and Lie algebra $\mathfrak{g}$

Let $G = \{[f] : f \in \mathcal{O}_0^\times\}$. Then $G$ can be regarded as a complex Lie group of infinite dimension.
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$$\langle h \rangle^m_n = \begin{cases} m [h]_{n-m+1}^1 & (m \leq n) \\ 0 & (m > n) \end{cases}$$

Then $\mathfrak{g} = \{ \langle h \rangle : h \in \mathcal{O}_0 \}$ can be identified with the Lie algebra of $G$. The Lie bracket is given by

$$[[h], [j]] = \langle h \rangle [j] - [j] \langle h \rangle.$$
We set $e_n = \langle z^{n+1} \rangle$ for $n \geq 0$. Then

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Thus $\mathfrak{g}$ can be regarded as the "positive part" of the Virasoro algebra with zero central charge.
Loewner ODE: \[
\frac{d}{dt} w_t = -w_t p_t(w_t).
\]

Loewner PDE: \[
\frac{d}{dt} f_t = z f_t' p_t.
\]
Matrix forms of Loewner equations

Matrix form (Schippers 2006)

$$\frac{d}{dt}[w_t] = -\langle zp_t \rangle [w_t].$$

$$\frac{d}{dt}[f_t] = [f_t] \langle zp_t \rangle$$