Loewner 方程式と SLE :
Loewner Equation and SLE : Overview

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2010 **年** 2 **月** 12 **日**
東北大学鳴子会館

Bieberbach conjecture

Bieberbach conjecture (1916)

Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be analytic and schlicht (univalent) in the unit disk $|z| < 1$. Then

$$|a_n| \leq n$$

for every n. Equality holds only if f is either the Koebe function $K(z) = \dfrac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots$ or its rotation $e^{-i\theta} K(e^{i\theta} z)$.

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- L. de Branges (1985) $|a_n| \leq n$ for all n

Loewner chain

Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be univalent in the unit disk $\mathbb{D} = \{|z| < 1\}$ and suppose that $\Omega = f(\mathbb{D})$ is a single-slit domain, namely, $\Omega = \mathbb{C} - \gamma([0, +\infty))$, where $\gamma : [0, \infty) \to \mathbb{C}$ is a simple arc with $\gamma(+\infty) = \infty$.

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Let $f_t : \mathbb{D} \to \mathbb{C} - \gamma([t, \infty))$ be the conformal homeomorphism determined by $f_t(0) = 0$ and $f_t'(0) > 0$. Note that $f_0 = f$ and thus $f_0'(0) = 1$. It is easy to see that $f_t'(0)$ is continuous and monotone increasing in $t \geq 0$. Changing the parameter if necessary, we may assume that $f_t'(0) = e^t$.

(original) Loewner PDE

Theorem (Loewner 1923)

Let $f_t(z)$ be as before. Then, there exists a continuous function $\zeta : [0, +\infty) \to \partial\mathbb{D}$ such that

$$\dot{f}_t(z) = z f_t'(z) \frac{\zeta(t) + z}{\zeta(t) - z}.$$

Here, $\dot{f}_t = \partial f_t / \partial t$.

This is the original form of the Loewner equation (or, more precisely, we should call it the Loewner PDE).

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This is the original form of the Loewner equation (or, more precisely, we should call it the Loewner PDE). Note that

$$f_t(\zeta(t)) = \gamma(t).$$

(original) Loewner ODE

By setting for a fixed $z \in \mathbb{D}$

$$w(t) = f_t^{-1}(f_0(z)), \quad t \geq 0,$$

we obtain the ODE

$$\dot{w}(t) = -w(t)\frac{\zeta(t) + w(t)}{\zeta(t) - w(t)}$$

with the initial condition

$$w(0) = z.$$

This is also called the Loewner equation (or, more precisely, we should call it the Loewner ODE).

Radial Loewner equation

Let $\gamma : (0, t_0) \to \mathbb{D} - \{0\}$ be a simple arc with $\gamma(0+) = 1$. Let $f_t : \mathbb{D} \to \mathbb{D} - \gamma((0, t])$ be the conformal homeomorphism determined by $f_t(0) = 0$, $f_t'(0) > 0$. This time, we can assume that $f_t'(0) = e^{-t}$. Then f_t satisfies the radial Loewner equation

$$\dot{f}_t(z) = -z f_t'(z) \frac{\zeta(t) + z}{\zeta(t) - z}$$

for a continuous function $\zeta(t)$ with $|\zeta(t)| = 1$.
Note also that

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By setting for a fixed $z \in \mathbb{D}$

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Let T_z be the lifetime of the solution to the above ODE. Then

$$\{z \in \mathbb{D} : T_z > t\} = \mathbb{D} - \gamma((0, t]).$$

Chordal Loewner equation

Denote by $\mathbb{H}$ the upper half-plane $\{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$. Let $\gamma : (0, t_0) \to \mathbb{H}$ be a simple arc with $\gamma(0+) \in \mathbb{R}$. Let $f_t : \mathbb{H} \to \mathbb{H} - \gamma((0, t])$ be the conformal homeomorphism with hydrodynamic normalization $f_t(z) = z + O(1/z)$ as $z \to \infty$ in $\mathbb{H}$. Let $f_t(z) = z + b(t)/z + O(1/z^2)$. Here, $b(t) > 0$ is a sort of capacity of $\gamma((0, t])$. This time, by a suitable re-parametrization, we can assume that $b(t) = 2t$. Then f_t satisfies the chordal Loewner equation

$$\dot{f}_t(z) = \frac{2 f_t'(z)}{U(t) - z}$$

for a continuous real-valued function $U(t)$.
Note also that

$$f_t(U(t)) = \gamma(t).$$

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SLE_κ

Let B_t be 1-dimensional Brownian motion and let $\kappa \geq 0$ be a constant.
Ths (chordal) SLE_κ is the random families of conformal maps g_t obtained as the solutions to the equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z.$$

(g_t is conformal homeomorphism of $\{z \in \mathbb{H} : T_z > t\}$ onto $\mathbb{H}$.)

Scaling limit of discrete processes

The following observation was made by Schramm:

Scaling limit of discrete processes

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If a model from statistical physics behaves in a conformally invariant way in the scaling limit, then the law of the interfaces converges to that of SLE_κ for a suitable κ.

Special cases

- $\kappa = 2$: loop-erased random walk and the uniform spanning tree
- $\kappa = 8/3$: self-avoiding random walk
- $\kappa = 3$: interfaces for the Ising model
- $\kappa = 4$: Gaussian free field
- $\kappa = 6$: percolation exploration process on the triangular lattice