Global construction formula for quasiconformal automorphisms of the plane

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1 Introduction

When I was a graduate student, I was recommended to read carefully the book "Lectures on quasiconformal mappings" written by professor Ahlfors. Every chapter of this book is quite interesting and specially I was fascinated with Chapter V, i.e., the existence problem of q.c. mappings with a given Beltrami diffrential.

Problem. *Let* $\mu \in L^\infty(\mathbb{C}) = L^\infty(\hat{\mathbb{C}})$ $(\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\})$ *with* $\|\mu\|_\infty < 1$. *Construct the homeomorphism* $f : \mathbb{C} \to \mathbb{C}$ *(or* $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *with* $f(\infty) = \infty$*) satisfying 3 conditions,*

(i) *distributional derivatives* $f_z, f_{\overline{z}} \in L^2_{loc}(\mathbb{C})$

(ii) $f_{\overline{z}} = \mu f_z$ *(Beltrami equation)*

(iii) $f(0) = 0, f(1) = 1, f(\infty) = \infty$.

As you konow that the unique solution exists. We denote it by

$$f^{\mu}.$$

Many matehmaticians tried to solve this problem and I do not know precisely who gave first a correct answer. However in the book, professor Ahlfors gave an elegant method of construction by making use of a singular integral operator, nowadays called Ahlfors-Beurling operator.

Let us recall the method of construction of f^{μ} by Ahlfors and Bers, which is described in the book. Then I can explain a motivation of this research work.

2 Ahlfors-Bers construction

We need two integral operators. First one is defined by

$$P_0 h(z) := \frac{1}{2\pi i} \int\int_{\mathbb{C}} \frac{z h(t)}{t(t-z)} \, dt \wedge d\bar{t}, \quad h \in L^p(\mathbb{C}),\ 2 < p < \infty.$$

Note that $dt \wedge d\bar{t} = -2i dt_1 dt_2$. For each $z \in \mathbb{C}$ the integarl in the right side of the equation converges. Second one is a so-called singular integral operator defined by

$$T_0 h(z) := \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int\int_{\mathbb{C}} \frac{h(t)}{(t-z)^2} \, dt \wedge d\bar{t}, \quad h \in L^p(\mathbb{C}),\ 1 < p < \infty.$$

Recently T_0 is called the Ahlfors-Beurling operator. For almost all $z \in \mathbb{C}$ the limit in the right side of the equation coverges. Operator T_0 is a bounded linear operator of $L^p(\mathbb{C})$ into itself. We have

$$C_p = \sup_{0 \neq h \in L^p(\mathbb{C})} \frac{\|T_0 h\|_p}{\|h\|_p} \to 1 \quad \text{as } p \to 2.$$

Operator P_0 and T_0 have nice properties. Equalities

$$(P_0h)_z = T_0h, \quad (P_0h)_{\overline{z}} = h \quad \text{for} \;\; h \in L^p(\mathbb{C}), \; 2 < p < \infty$$

hold in the sense of distribution.

Suppose that the solution $f = f^\mu$ can be expressed in a form

$$f(z) = z + P_0h(z)$$

for some $h \in L^p(\mathbb{C}), \; 2 < p < \infty$. Then we have

$$f_z = 1 + T_0h, \quad f_{\overline{z}} = h$$

in the sense of distribution. Thus we have

$$h = f_{\overline{z}} = \mu f_z = \mu(1 + T_0h)$$

and hence

$$(1 - \mu T_0)h = \mu.$$

For the moment we suppose

$$\|\mu\|_\infty < \frac{1}{C_p}$$

(since $C_p \to 1$ as $p \to 2$, this holds for all p sufficiently close to 2)

and

$$\mu \in L^p(\mathbb{C}) \;\; \text{for some}\, p \in (2, \infty).$$

Then since

the operator norm $\|\mu T_0\| \leq \|\mu\|_\infty \|T_0\| = \|\mu\|_\infty C_p < 1$,

the equation

$$(1 - \mu T_0)h = \mu$$

can be invertible by the Neumann series expansion,

$$f_{\overline{z}} = h = \sum_{n=0}^{\infty} (\mu T_0)^n \mu$$

and hence

$$f(z) = z + P_0 h(z) = z + \sum_{n=0}^{\infty} P_0((\mu T_0)^n \mu)(z).$$

Also we can prove f is a automorphism of $\hat{\mathbb{C}}$ satisfies all of the desired properties except

$$f(1) = 1.$$

By noramlization we have

Theorem A If $\mu \in L^\infty(\mathbb{C})$ satisfies

$$\|\mu\|_\infty < \frac{1}{C_p} \text{ and } \mu \in L^p(\mathbb{C})$$

for some $p \in (2, \infty)$, then

$$f^\mu(z) = \frac{z + \sum_{n=0}^{\infty} P_0((\mu T_0)^n \mu)(z)}{1 + \sum_{n=0}^{\infty} P_0((\mu T_0)^n \mu)(1)}.$$

By making use of the above theorem, we can construct f^μ as follows,

1 Suppose suppμ is compact. In this case since $\mu \in L^p(\mathbb{C})$ for all $p \in (1, \infty)$, we can construct f^μ by the above theorem.

2 Suppose $0 \notin$ suppμ. In this case put

$$\tilde{\mu}(z) := \left(\frac{z}{\overline{z}}\right)^2 \mu\left(\frac{1}{z}\right).$$

Then supp$\tilde{\mu}$ is compact and we can construct $f^{\tilde{\mu}}$. Also we can prove

$$f^{\mu}(z) = \frac{1}{f^{\tilde{\mu}}(1/z)}.$$

by calculating the Beltrami coefficient of the right hand side of the equation.

3 For general μ we decompose

$$\mu = \mu_1 + \mu_2, \quad \text{supp}\mu_1 \text{ is compact and } 0 \notin \text{supp}\mu_2.$$

Put

$$\lambda = \frac{\mu - \mu_2}{1 - \mu\overline{\mu_2}} \frac{f_z^{\mu_2}}{\overline{f_z^{\mu_2}}} \circ (f^{\mu_2})^{-1}.$$

Then $\|\lambda\|_\infty < 1$ and suppλ is compact. Finally we can prove

$$f^{\mu} = f^{\lambda} \circ (f^{\mu_2})^{-1}$$

by calculating its Beltrami coefficient.

The method of construction of f^{μ} is elegant, however

(i) Mapping f^{μ} is an automorphism of $\hat{\mathbb{C}}$. But the construction employ $L^p(\mathbb{C})$ and T_0 and P_0. These are adapted to the complex plane $\mathbb{C}$, not to $\hat{\mathbb{C}}$.

(ii) Construction for general μ is too complicated. This casuse a big difficulty on finding a variational formula of general order for q.c. mappings. Later we will give a typical variational formula for q.c. mappings as an application of our result.

Since f^{μ} is an automorphism of $\hat{\mathbb{C}}$, I belive that a more global result must exist.

3 A variant of Ahlfors-Beurling operator

Strategy

We want to find

$$\begin{aligned}&\mathcal{B}: \text{ a Banach space of functions in } \hat{\mathbb{C}} \text{ with } L^\infty(\mathbb{C}) \subset \mathcal{B}\\ &T: \mathcal{B} \to \mathcal{B}, \text{ linear and bounded}\\ &P: \text{ an opertor acts on } \mathcal{B}\end{aligned}$$

satisfying

$$(Ph)_z = Th, \quad (Ph)_{\overline{z}} = h \quad \text{in the sense of distribution}$$

and

$$Ph(0) = Ph(1) = 0\ \forall h \in \mathcal{B}.$$

If we can find $\mathcal{B}$, T and P, then probably we can prove

$$f^\mu_{\overline{z}} = \sum_{n=0}^\infty (\mu T)^n \mu$$

$$f^\mu(z) = z + \sum_{n=0}^\infty P(\mu T)^n \mu(z)$$

for

$$\mu \in L^\infty(\mathbb{C}) \text{ with } \|\mu\|_\infty < \frac{1}{\|T\|}.$$

No assumption on $\operatorname{supp}\mu$ is necessary, however we need an extra condition $\|\mu\|_\infty < \|T\|^{-1}$. To find $\mathcal{B}$, T, P we start with the following,

Theorem 1 *If f is a quasiconformal automorphism of $\hat{\mathbb{C}}$ normalized by $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$, and its Beltrmai coeffcient μ satisfies $\|\mu\|_\infty < 1/3$, then*

$$(1) \qquad f(z) = z + \frac{1}{2\pi i}\int\int_{\mathbb{C}} \frac{z(z-1)f_{\overline{z}}(t)}{t(t-1)(t-z)} dt \wedge d\overline{t}.$$

This can be derived from the Pompeiu formula

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i}\int\int_{|\zeta|<1} \frac{f_{\overline{\zeta}}(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta},$$

and $f(0) = 0$, $f(1) = 1$ $f(\infty) = \infty$. The condition $\|\mu\|_\infty < 1/3$ guarantees absolute convergence of the integral in (1)

The foumula (1) suggests a new oprator

$$\begin{aligned} Ph(z) &:= \frac{1}{2\pi i}\int\int_{\mathbb{C}} \frac{z(z-1)h(t)}{t(t-1)(t-z)} dt \wedge d\overline{t} \\ &= P_0h(z) - \frac{1}{2\pi i}\int\int_{\mathbb{C}} \frac{zh(t)}{t(t-1)} dt \wedge d\overline{t}. \end{aligned}$$

Then we have

$$\begin{aligned} &(Ph)_{\overline{z}} = (Ph)_{\overline{z}} = h \\ &(Ph)_z = T_0h - \frac{1}{2\pi i}\int\int_{\mathbb{C}} \frac{h(t)}{t(t-1)} dt \wedge d\overline{t}. \end{aligned}$$

So we put

$$Th(z) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i}\int\int_{|t-z|>\varepsilon} \left\{ \frac{1}{(t-z)^2} - \frac{1}{t(t-1)} \right\} h(t)\, dt \wedge d\overline{t}.$$

Then we have

$$(Ph)_z = Th.$$

Thus we could get good candidates of T and P. How about $\mathcal{B}$ adapted to Riemann sphere $\hat{\mathbb{C}}$? The easiest and natural candidate is

$$\begin{aligned} S^p(\hat{\mathbb{C}}) \; &:= \; \text{the space of all Lebesgue measurable functions } h \text{ on } \hat{\mathbb{C}} \\ & \qquad \text{with } \|h\|^p_{S^p(\hat{\mathbb{C}})} = \frac{1}{\pi}\int\int_{\mathbb{C}} \frac{|h(z)|}{(1+|z|^2)^2}\, dx \wedge dy < \infty. \end{aligned}$$

Now we can state our theorems.

Theorem 2 *Let* $p \in (2,\infty)$ *and* $h \in S^p(\hat{\mathbb{C}})$. *Then*

(i) *Ph is Hölder continuous in* $\mathbb{C}$.

(ii) *$Th(z)$ exists for almost all $z \in \mathbb{C}$ and $Th \in S^p(\hat{\mathbb{C}})$. Furthermore*

$$T : S^p(\hat{\mathbb{C}}) \to S^p(\hat{\mathbb{C}}) \text{ is bounded and linear}$$

(3) *The equation*

$$(Ph)_z = Th, \quad (Ph)_{\overline{z}} = h$$

hold in the sense of distribution.

Put

$$D_p = \sup_{0 \neq h \in S^p(\hat{\mathbb{C}})} \frac{\|Th\|_{S^p(\hat{\mathbb{C}})}}{\|h\|_{S^p(\hat{\mathbb{C}})}}, \quad p \in (2,\infty).$$

Then

Theorem 3 *Let* $p \in (2,\infty)$. *Then for $\mu \in L^\infty(\mathbb{C})$ with* $\|\mu\|_\infty < \min\{1/3, 1/D_p\}$ *the expansion*

$$f^\mu_{\overline{z}} = \sum_{n=0}^{\infty} (\mu T)^n \mu$$

holds, where the series converges absolutely in $S^p(\hat{\mathbb{C}})$. Furthermore the expansion

$$f^\mu(z) = z + \sum_{n=0}^{\infty} P(\mu T)^n \mu(z)$$

hold for each fixed $z \in \mathbb{C}$.

4 Boundedness of the oprator T

It suffices to show,

Proposition 4 *There exists a constant $C(p)$ depending only on $p \in (2, \infty)$ such that*

$$\|Th\|_{S^p(\mathbb{C})} \le C(p)\|h\|_{S^p(\mathbb{C})}, \quad \forall h \in C_c^2(\mathbb{C}^*), \tag{2}$$

where $C_c^2(\mathbb{C}^*)$ is the space of all C^2-functions h on $\mathbb{C}$ with $h(0) = h(1) = 0$.

Sketch of Proof. For $|z| \le 1$ we have

$$\begin{aligned}
& Th(z) \\
= \; & \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int\int_{|t-z|<\varepsilon,\, |t|<2} \{\frac{1}{(t-z)^2} - \frac{1}{t(t-1)}\} h(t)\, dt \wedge d\bar{t} \\
& + \frac{1}{2\pi i} \int\int_{|t|>2} \{\frac{1}{(t-z)^2} - \frac{1}{t(t-1)}\} h(t)\, dt \wedge d\bar{t} \\
= \; & T_0(\chi_{\mathbb{D}(2)} h)(z) - \frac{1}{2\pi i} \int\int_{|t|<2} \frac{h(t)}{t(t-1)} dt \wedge d\bar{t} + T((1-\chi_{\mathbb{D}(2)})h)(z).
\end{aligned}$$

We estimate $S^p(\mathbb{D})$ norm of each term in the right hand side of the above inequality. First we have

$$\begin{aligned}
& \|T_0(\chi_{\mathbb{D}(2)} h)\|^p_{S^p(\mathbb{D})} \\
= \; & \frac{1}{\pi} \int\int_{|z| \le 1} \frac{|T_0(\chi_{\mathbb{D}(2)} h)(z)|^p}{(1+|z|^2))^2}\, dm(z) \\
\le \; & \frac{1}{\pi} \int\int_{|z| \le 1} |T_0(\chi_{\mathbb{D}(2)} h)(z)|^p\, dm(z) \\
\le \; & \frac{1}{\pi} \int\int_{\mathbb{C}} |T_0(\chi_{\mathbb{D}(2)} h)(z)|^p\, dm(z)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\pi}\|T_0(\chi_{\mathbb{D}(2)}h)\|^p_{L^p(\mathbb{C})}\\
&\le \frac{C_p^p}{\pi}\|\chi_{\mathbb{D}(2)}h\|^p_{L^p(\mathbb{C})}\\
&\le \frac{C_p^p}{\pi}\int\int_{|t|<2}|h(t)|^p\,dm(t)\\
&\le \frac{25C_p^p}{\pi}\int\int_{|t|<2}\frac{|h(t)|^p}{(1+|t|^2)^2}\,dm(t)\\
&\le 25C_p^p\|h\|^p_{S^p(\mathbb{C})}
\end{aligned}
$$

Next we have

$$
\begin{aligned}
&|T((1-\chi_{\mathbb{D}(2)})h)(z)|\\
\le\ & \frac{1}{\pi}\int\int_{|t|>2}\left|\frac{1}{(t-z)^2}-\frac{1}{t(t-1)}\right||h(t)|\,dm(t)\\
\le\ & \frac{1}{\pi}\int\int_{|t|>2}\frac{|2zt-z^2-t|}{|(t-z)^2t(t-1)|}|h(t)|\,dm(t)\\
\le\ & \frac{1}{\pi}\int\int_{|t|>2}\frac{|2z-1||t|+|z|^2}{(|t|-|z|)^2|t||t-1||}|h(t)|\,dm(t)\\
\le\ & \frac{1}{\pi}\int\int_{|t|>2}\frac{3|t|+1}{|t|(|t|-1)^3}|h(t)|\,dm(t)\\
\le\ & \frac{1}{\pi}\int\int_{|t|>2}\frac{(3|t|+1)(1+|t|^2)^{2/p}}{|t|(|t|-1)^3}\frac{|h(t)|}{(1+|t|^2)^{2/p}}\,dm(t)\\
\le\ & \left\{\frac{1}{\pi}\int\int_{|t|>2}\frac{(3|t|+1)(1+|t|^2)^{2/p}}{|t|(|t|-1)^3}\right\}^{1/q}\cdot\left\{\frac{1}{\pi}\int\int_{|t|>2}\frac{|h(t)|^p}{(1+|t|^2)^2}\,dm(t)\right\}^{1/p}\\
\le\ & \text{const.}\|h\|_{S^p(\mathbb{C})}.
\end{aligned}
$$

Thus we have

$$
\|T((1-\chi_{\mathbb{D}(2)})h)\|_{S^p(\mathbb{D})}\le \text{const.}\|h\|_{S^p(\mathbb{C})}.
$$

Similarly we have

$$(3) \qquad \left|\frac{1}{2\pi i}\int\int_{|t|<2}\frac{h(t)}{t(t-1)}\right| \le \text{const.}\|h\|_{S^p(\mathbb{C})}.$$

Combining these inequalities we have

$$\|Th\|_{S^p(\mathbb{D})} \le \text{const.}\|h\|_{S^p(\mathbb{C})}.$$

For $|z| > 1$ we have

$$Th(z) = T\tilde{h}(1/z) - 2zP\tilde{h}(1/z),$$

where

$$\tilde{h}(z) = \left(\frac{z}{\bar{z}}\right)^2 h(1/z).$$

Noting $\|\tilde{h}\|_{S^p(\mathbb{D})} = \|h\|_{S^p(\mathbb{C}\setminus\mathbb{D})}$ we have

$$\|T\tilde{h}(1/z)\|_{S^p(\mathbb{C}\setminus\mathbb{D})} = \|T\tilde{h}(z)\|_{S^p(\mathbb{D})} \le \text{const.}\|\tilde{h}\|_{S^p(\mathbb{C})} = \text{const.}\|h\|_{S^p(\mathbb{C})}.$$

Next we have $\|zP\tilde{h}(1/z)\|_{S^p(\mathbb{C}\setminus\mathbb{D})} = \|z^{-1}P\tilde{h}(z)\|_{S^p(\mathbb{D})}$ and

$$
\begin{aligned}
&\frac{1}{z}P\tilde{h}(z) \\
= \;& \frac{1}{2\pi i}\int\int_{|t|<2}\frac{\tilde{h}(t)}{t(t-z)}\,dt\wedge d\bar{t} - \frac{1}{2\pi i}\int\int_{|t|<2}\frac{\tilde{h}(t)}{t(t-1)}\,dt\wedge d\bar{t} \\
&+\frac{1}{2\pi i}\int\int_{|t|>2}\frac{(z-1)\tilde{h}(t)}{t(t-1)(t-z)}\,dt\wedge d\bar{t} \\
= \;& \frac{P_0(\chi_{\mathbb{D}(2)}\tilde{h})(z) - P_0(\chi_{\mathbb{D}(2)}\tilde{h})(0)}{z} + A + B(z) \quad \text{(say).}
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
|A| &\le \text{const.}\|\tilde{h}\|_{S^p(\mathbb{C})} = \text{const.}\|h\|_{S^p(\mathbb{C})} \\
\|B\|_{S^p(\mathbb{D})} &\le \text{const.}\|\tilde{h}\|_{S^p(\mathbb{C})} = \text{const.}\|h\|_{S^p(\mathbb{C})}.
\end{aligned}
$$

To estimate the 1st term we need a lemma on Sobolev functions,

Lemma 5 *For $u \in W^{1,p}(\mathbb{C})$ with $p \in (2,\infty)$ we have*

$$
\left\{\int\int_{\mathbb{C}}\left|\frac{u(z)-u(0)}{z}\right|\right\}^{1/p} \le \frac{p}{p-2}\left\{\|u_z\|_{L^p(\mathbb{C})} + \|u_{\bar{z}}\|_{L^p(\mathbb{C})}\right\}.
$$

Applying the lemma to $u(z) = P_0(\chi_{\mathbb{D}(2)}\tilde{h})(z)$ we get

$$
\begin{aligned}
&\left\|\frac{P_0(\chi_{\mathbb{D}(2)}\tilde{h})(z) - P_0(\chi_{\mathbb{D}(2)}\tilde{h})(0)}{z}\right\|_{S^p(\mathbb{D})} \\
\le \;& \frac{p}{p-2}\left\{\|P_0(\chi_{\mathbb{D}(2)}\tilde{h})_z\|_{L^p(\mathbb{C})} + \|P_0(\chi_{\mathbb{D}(2)}\tilde{h})_{\bar{z}}\|_{L^p(\mathbb{C})}\right\} \\
= \;& \frac{p}{p-2}\left\{\|T_0(\chi_{\mathbb{D}(2)}\tilde{h})\|_{L^p(\mathbb{C})} + \|\chi_{\mathbb{D}(2)}\tilde{h}\|_{L^p(\mathbb{C})}\right\} \\
\le \;& \frac{p}{p-2}\left\{C_p\|\chi_{\mathbb{D}(2)}\tilde{h}\|_{L^p(\mathbb{C})} + \|\chi_{\mathbb{D}(2)}\tilde{h}\|_{L^p(\mathbb{C})}\right\} \\
\le \;& \text{const.}\|\tilde{h}\|_{S^p(\mathbb{C})} = \text{const.}\|h\|_{S^p(\mathbb{C})}.
\end{aligned}
$$

Combining these inequalities we have

$$(4)\quad \begin{aligned} \|Th(z)\|_{S^p(\mathbb{C}\setminus\mathbb{D})} &= \|T\tilde{h}(1/z) - 2zP\tilde{h}(1/z)\|_{S^p(\mathbb{C}\setminus\mathbb{D})} \\ &= \|T\tilde{h}(z) - 2z^{-1}P\tilde{h}(z)\|_{S^p(\mathbb{D})} \\ &\le \|T\tilde{h}\|_{S^p(\mathbb{D})} + 2\|z^{-1}P\tilde{h}\|_{S^p(\mathbb{D})} \le \text{const.}\|h\|_{S^p(\mathbb{C})} \end{aligned}$$

Finally we have by (3) and (4)

$$(5)\quad \|Th\|_{S^p(\mathbb{C})} \le \|Th\|_{S^p(\mathbb{D})} + \|Th\|_{S^p(\mathbb{C}\setminus\mathbb{D})} \le \text{const.}\|h\|_{S^p(\mathbb{C})}$$

5 Application on variational formula for qasiconfromal mappings

Suppose that $\mu_t = t\mu_1 + t^2\mu_2 + \cdots$, where the series converges absolutely in $L^\infty(\mathbb{C}) = S^\infty(\hat{\mathbb{C}})$. The Ahlfors-Bers theorem asserts that if μ_t varies holomorphically on t, then f^{μ_t} varies also holomorphically on t. Thus we have

$$(6)\qquad f^{\mu_t}(z) = z + tA_1(z) + t^2A_2(z) + \cdots.$$

Since we have from Theorem 3

$$\begin{aligned} f^{\mu_t}(z) &= z + P\mu_t(z) + P(\mu_t T)\mu_t(z) + \cdots \\ &= z + tP\mu_1(z) + t^2P\mu_2(z) + t^2P\mu_1T\mu_1 + O(t^3). \end{aligned}$$

We can easily have

$$\begin{aligned} A_1(z) &= P\mu_1(z) \\ A_2(z) &= P\mu_2(z) + P\mu_1T\mu_1(z), \\ &\vdots\ . \end{aligned}$$

Of course we can also calculate higher terms.

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