QUASICONFORMAL EXTENSIONS OF A CIRCLE-HOMEOMORPHISM

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Letting Δ_z and Δ_w denote the unit closed disk in the z- and w-plane respectively, we consider an orientation- preserving homeomorphism $\gamma(\theta)$ of $\partial\Delta_z$ onto $\partial\Delta_w$. Some $\gamma(\theta)$ may admit extension up to quasiconformal homeomorphisms of $\operatorname{Int}\Delta_z$ onto $\operatorname{Int}\Delta_w$, the totality of which shall be denoted by Γ_{Q}. On the other hand, some $\gamma(\theta)$ may be extensible up to an ACL^2-mapping $f(z)$ of $\operatorname{Int}\Delta_z$ onto $\operatorname{Int}\Delta_w$, whose Dirichlet integral

$$E[f] = \iint_{\operatorname{Int}\Delta_z} [|f_z|^2 + |f_{\overline{z}}|^2] dxdy \qquad (z = x + iy)$$

is finite, totality of which shall be denoted by Γ_{E}. By the way we denote such extension of γ by $\mathrm{Dxt}[\gamma]$.

Proposition 1.

$$\Gamma_{\mathrm{Q}} \subseteq \Gamma_{\mathrm{E}}.$$

Proof. Taking an arbitrary element $\gamma(\theta)$ out of Γ_{Q}, let us denote by $f(z)$ a q. c. extension of $\gamma(\theta)$. Note that $f^{-1}(w)$ is K-quasiconformal in $\operatorname{Int}\Delta_w$ under the assumption that $f(z)$ is K-quasiconformal in $\operatorname{Int}\Delta_z$. Hence for the dilatation $Q_f(z)$ of $f(z)$ we have

$$\begin{aligned} \frac{\pi}{2}\left(K_f + \frac{1}{K_f}\right) &\geq \frac{1}{2}\iint_{\operatorname{Int}\Delta_w} [Q_{f^{-1}}(w) + (1/Q_{f^{-1}}(w))] dudv \qquad (w = u + iv) \\ &= \iint_{\operatorname{Int}\Delta_z} [|p_f(z)|^2 + |q_f(z)|^2] dxdy, \end{aligned} \tag{1}$$

where K_f stands for the maximal dilatation in the interior of the disk. □

Lemma 1. *Suppose that a real-valued periodic function $\varphi(\theta)$ is defined on the θ-interval $I = [0, 2\pi]$ so as to be $\varphi(0) = \varphi(2\pi)$ satisfying the Lipschitz condition*

$$|\varphi(\theta_1) - \varphi(\theta_2)| \leq |\theta_1 - \theta_2|$$

on I. Then for an arbitrary positive constant $c < 1$, the function $\Phi(\theta) = \theta + c\varphi(\theta)$ is a strongly monotone-increasing function on I which belongs to the class Lip *1.*

Theorem 1. *There exists at least one orientation-preserving homeomorphism $\gamma(\theta)$ of $\partial\Delta_z$ onto $\partial\Delta_w$ which posesses the following properties:*

(1°) $\gamma(\theta)$ is prolongable up to an ACL^2 *orientation-preserving continuous map $f(z)$ of* $\operatorname{Int}\Delta_z$ *onto* $\operatorname{Int}\Delta_w$.

(2°) $f(z)$ has finite Dirichlet integral over $\operatorname{Int}\Delta_z$.

Proof. If we set $\gamma(\theta) = \exp\{i\Phi(\theta)\}$, then $\gamma(\theta)$ provides a homeomorphism of $\partial\Delta_z$ onto $\partial\Delta_w$ (Lemma 1). Abel mean

$$f(re^{i\theta}) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)r^n$$

Date: (Read on 8th January, 2003).

Key words and phrases. Dirichlet integral, Teichmüller map.

2000 *Subject Classification.* Primary 30C70. Secondary 31A25.

of Fourier expansion

$$\gamma(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

is harmonic in $\operatorname{Int}\Delta_z$ and Radó-Kneser-Choquet's theorem together with Lemma 1 assure that $f(z)$ maps Δ_z topologically onto Δ_w. Since $n|a_n| \leq 2\pi, n|b_n| \leq 2\pi (n = 1, 2, ...)$ and $\sum_{n=1}^{\infty}(|a_n| + |b_n|) < +\infty$, we have finally

$$E[f(z)] = \pi \sum_{n=1}^{\infty} n(|a_n|^2 + |b_n|^2) < +\infty. \quad \square$$

Now let γ be an arbitrary element of Γ_{E}. The family $\mathrm{Dxt}[\gamma]$ is normal and any minimizing sequence for the functional $E[f]$ contains a subsequence $\{f_n(z)\}$ uniformly convergent on Δ_z: Put $f^*(z) = \lim_{n\to\infty} f_n(z)$. We must have

$$\begin{aligned} E[f^*] &= \lim_{n\to\infty} E[f_n] \\ &= \inf_{f \in \mathrm{Dxt}[\gamma]} E[f] \end{aligned}$$

in virtue of the lower semi-continuity of $\mathrm{Dxt}[\gamma]$ together with the compactness of $\mathrm{Dxt}[\gamma]$ in the topology of uniform convergence on Δ_z. Let $\mathcal{H}$ denote the aggregate of C^{∞}-function $h(z)$ vanishing outside the open disk $\operatorname{Int}\Delta_z$.

Definition 1 (Gerstenhaber-Rauch's variation)**.** We call

$$E[f^*(z + \varepsilon h(z))] - E[f^*(z)]$$

to be *Gerstenhaber-Rauch's variation* of the Dirichlet energy functional $F[f(z)]$ at $f = f^*$.

The Gerstenhaber-Rauch variation of $E[f]$ at $f = f^*$ referring to $h(z)$ of $\mathcal{H}$ reads

$$E[\tilde{f^*}] - E[f^*] = 4\mathrm{Re}\{\varepsilon \iint_{\operatorname{Int}\Delta_z} p_{f^*}\overline{q_{f^*}} h_{\overline{z}} dxdy\} + o(\varepsilon).$$

Since $E[f(z)]$ stagnates at $f = f^*$, Green's theorem provides us with the following relations:

$$\begin{aligned} 0 = \iint_{\operatorname{Int}\Delta_z} p_{f^*}\overline{q_{f^*}} h_{\overline{z}} dxdy &= \oint_{\partial\Delta_z} p_{f^*}\overline{q_{f^*}} h(z)dz - \iint_{\operatorname{Int}\Delta_z} [p_{f^*}\overline{q_{f^*}}]_{\overline{z}}\, h(z) dxdy \\ = -\iint_{\operatorname{Int}\Delta_z} [p_{f^*}\overline{q_{f^*}}]_{\overline{z}}\, h(z) dxdy. \end{aligned}$$

In view of arbitrariness of $h(z)$ in the interior to Δ_z we must have

$$\frac{\partial}{\partial \overline{z}} p_{f^*}(z)\overline{q^*(z)} \equiv 0 \qquad \text{in } \operatorname{Int}\Delta_z.$$

Remark 1. So far as the present questions on Dirichlet functional $E[f(z)]$ is concerned, the Gerstenhaber-Rauch variation

$$E[f(z + \varepsilon h(z))] - E[f(z)]$$

vanishes if and only if the Euler-Lagrange variation

$$E[f(z) + \varepsilon h(z)] - E[f(z)]$$

vanishes.

Theorem 2. *There exists a harmonic homeomorphism $f^*(z)$ belonging to* $\mathrm{Dxt}[\gamma]$.

Proof. The harmonic mapping $f^*(z)$ obtained above gives a homeomorphism between the disks in question.

Corollary 1. *The quadratic differential $\omega_{f^*}(z) = p_{f^*}(z)\overline{q_{f^*}(z)}dz^2$ is holomorphic in* $\operatorname{Int}\Delta_z$.

Suppose that two members of the trajectories $\mathcal{T} = \{\omega_{f^*} > 0\}$ as well as two members of the orthogonal trajectories $\mathcal{O} = \{\omega_{f^*} < 0\}$ determine a certain curvilinear quadrilateral Ω_z in $\operatorname{Int}\Delta_z$ whose conformal image shall be the rectangle

$$R_z = \{(x, y)|\ 0 \le x \le a, 0 \le y \le b\}.$$

In a similar manner let a univalent holomorphic function $w = \psi(\zeta)$ map the rectangle

$$R_\zeta = \{(\xi, \eta)|\ 0 \le \xi \le \alpha, 0 \le \eta \le \beta\}$$

in the $\zeta = \xi + i\eta$-plane onto the curvilinear quadrilateral $\Omega_w = f^*(\Omega_z)$ so that the horizontal and vertical sides of R_z and R_ζ may correspond to each other.

Let $\mathcal{S}$ denote the subclass of $\mathrm{Dxt}[\gamma]$ that sends Ω_z onto Ω_w. Then we are allowed to adopt the following identifications: The element $S(z)$ of the class $\mathcal{S}$ is a continuous mapping of R_z onto R_ζ which is ACL^2 in its interior, and we write

$$E^*[S(z)] = \iint_{R_z} |\psi'(\zeta)|^2[|p_S|^2 + |q_S|^2]dxdy.$$

in terms of *weighted* Dirichlet integral of the rectangular mapping $S(z)$.

Proposition 2. *The restriciton $S_0(z)$ of the mapping $f^*(z)$ to Ω_z minimizes the funcitonal*

$$E^*[S(z)] = \iint_{\Omega_z} |\psi'(\zeta)|^2[|p_S|^2 + |q_S|^2]dxdy$$

within the subfamily $\mathcal{S}$.

Proposition 3. *The family $\mathcal{S}$ is convex.*

Proof. Indeed, for any $S_j(z)$ $(j = 1, 2)$ of $\mathcal{S}$ and a real parameter t ranging over the interval $[0, 1]$, the linear combination $tS_1(z) + (1 - t)S_2(z)$ belongs to $\mathcal{S}$.

Definition 2. $E^*[S(z)]$ defined on $\mathcal{S}$ is said to be *stationary* at $S_0(z)$ if the Euler-Lagrange's first variation of $E^*[S(z)]$ vanishes at $S = S_0$.

Theorem 3. *If $E^*[S(z)]$ is stationary at one point $S = S_0$ of $\mathcal{S}$, then S_0 gives a local minimum to $E^*[S(z)]$ on $\mathcal{S}$.*

Proof. Take an arbitrary element $S(z)$ of $\mathcal{S}$. For any real parameter t ranging over the interval $(0, 1), \tilde{S}_0(z) = (1 - t)S_0(z) + tS(z)$ belongs to $\mathcal{S}$ (Proposition 3). For any $t \in (0, 1)$ we have

$$\begin{aligned}
&\frac{E^*[\tilde{S_0}] - E^*[S_0]}{t}\\
&= (2 + t)E^*[S_0] + tE^*[S] + 2\iint_R \mathrm{Re}\{\mathrm{p_{S_0}}\overline{\mathrm{p_S}} + \mathrm{q_{S_0}}\overline{\mathrm{q_S}}\}|\psi'(\zeta)|^2\mathrm{dxdy}\\
&\le (-2 + t)E^*[S_0] + tE^*[S] + 2\iint_R (|p_{S_0}||p_S| + |q_{S_0}||q_S|)|\psi'(\zeta)|^2dxdy.
\end{aligned}$$

Letting $t \ne 0$ tend to 0, we obtain by assumtion that

$$\begin{aligned}
0 &= \left.\frac{dE^*[S_t(z)]}{dt}\right|_{t=0}\\
&\le -2E^*[S_0] + 2\iint_R \sqrt{(|p_{S_0}|^2 + |q_{S_0}|^2)(|p_S|^2 + |q_S|^2)}|\psi'(\zeta)|^2dxdy\\
&\le -2E^*[S_0] + 2\sqrt{E^*[S_0]E^*[S]}.
\end{aligned} \tag{2}$$

Hence

$$E^*[S_0] \le E^*[S],$$

and the equality sign in the last relation enters only when $S(z) \equiv S_0(z)$. In other words, $S(z) \ne S_0(z)$ implies $E^*[S] > E^*[S_0]$. □

Definition 3. Let $\mathcal{H}_1$ denote the subfamily of $\mathcal{H}$ which vanishes identically outside Ω_z.

Definition 4 (harmonic mapping)**.** An element $S(z)$ of $\mathcal{S}$ is called *harmonic* relative to the conformal metric $ds^2 = |\psi'(\zeta)|^2|d\zeta|^2$ iff the quadratic differential $\tilde{\omega}_S = |\psi'(\zeta)|^2 p_S(z)\overline{q_S(z)}dz^2$ is holomorphic in $\operatorname{Int} R_z$.

Definition 5. If an ACL^2 topological mapping $f(z)$ defined in a neighbourhood U on the z-plane is expressed in a composite form *conformal* $\circ$ *affine* $\circ$ *conformal*, so $f(z)$ is called to be *locally Teichmüller* in U.

Proposition 4. *Any locally Teichmüller mapping is harmonic.*

Proof. Let $f(z)$ be locally Teichmüller. Then the quadratic differential $\omega_f = p_f(z)\overline{q_f(z)}dz^2$ is holomorphic. □

Proposition 5. *The affine linear mapping $\zeta = T(z)$ of R_z onto R_ζ endows the weighted Dirichlet integral $E^*[S(z)]$ with its absolute minimum in the family $\mathcal{S}$.*

Proof. Since the mapping $\psi \circ T(z)$ is locally Teichmüller in the interior to R_z by definition, its characteristic arcs coïncide with the trajectories of the holomorphic quadratic differential $\tilde{\omega}_T = |\psi'(\zeta)|^2 p_T(z)\overline{q_T(z)}dz^2$. Operating the Gerstenhaber-Rauch variation $z \mapsto \zeta = z + \varepsilon h(z)$ with $h(z)$ belonging to $\mathcal{H}_1$ to obtain $\tilde{T}(z) = T(\tilde{z})$, we have

$$E^*[\tilde{T}(z)] - E^*[T(z)] = \operatorname{Re}\{\varepsilon \iint_{R_z} [|\psi'(\zeta)|^2 p_T(z)\overline{q_T(z)}]h_{\overline{z}}dxdy\} + o(\varepsilon).$$

It follows with the aid of the boundary behaviour of $h(z)$ that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{E^*[\tilde{T}(z)] - E^*[T(z)]}{\varepsilon} &= -\iint_{R_z}[|\psi'(z)|^2 p_T(z)\overline{q_T(z)}]_{\overline{z}}h(z)dxdy \\ &= 0,\end{aligned}$$

which shows that the weighted Dirichlet integral $E^*[S(z)]$ stagnates at $S(z) = T(z)$. Therefore $T(z)$ attains the minimum $E^*[T(z)]$ on $\mathcal{S}$ (Theorem 5). □

Theorem 4. *If the weighted Dirichlet integral $E^*[S(z)]$ stagnates at some point $S_0(z)$ of the family $\mathcal{S}$, then it holds that $S_0(z) = T(z)$.*

Proof. First note that we have $E^*[S(z)] > E^*[S_0]$ for all $S(z) \neq S_0(z)$ (Proposition 4). Hence $S_0 \neq T(z)$ implies $E^*[T(z)] > E^*[S_0]$, while the Proposition 5 asserts that $E^*[T(z)] \leq E^*[S_0]$ leading to a contradiction.

Our next aim is to prove the inverse inclusion to Propositon 1. To this end we fix an arbitrary element γ of Γ_{E} and take an element $f(z)$ of $\operatorname{Axt}[\gamma]$ such that $E[f] < +\infty$. Then we shall agree to say that the f belongs to the class $\operatorname{Dxt}[\gamma]$. Letting $h(z)$ be an ACL^2-function vanishing on $\partial\Delta_z$, we shall distort f into $\tilde{f}(z) = f(z + \varepsilon h(z))$. Under the assumption that some $f^*(z)$ of $\operatorname{Dxt}[\gamma]$ is a stationary point for $E[f]$, we have

$$o(\varepsilon) = 4\operatorname{Re}\left\{\varepsilon \iint_{\mathrm{Int}\,\Delta_z} \mathrm{p_{f^*}}\overline{\mathrm{q_{f^*}}}\mathrm{h}_{\overline{\mathrm{z}}}\right\}\mathrm{dxdy}.$$

Green's theorem affords that

$$\iint_{\mathrm{Int}\,\Delta_z}(p_{f^*}\overline{q_{f^*}})_{\overline{z}}h(z)dxdy = 0. \tag{3}$$

The last relation (4) shows holomorphy of the quadratic differential $\omega_{f^*} = p_{f^*}\overline{q_{f^*}}dz^2$, what amounts to the same thing to say that the mapping $f^*(z)$ is harmonic in the interior of Δ_z.

Back to the pair of curvilinear quadrilaterals Ω_z and Ω_w we see that $f^*(z)$ maps Ω_z onto Ω_w quasiconformally with a constant dilatation.

Suppose the circumstance that the interior of two characteristic quadrilaterals Ω and Ω' like Ω_z is not disjoint with one another. Then we see that the constant dillatation of $f^*(z)$ in these two quadrilaterals Ω and Ω' shares the value K. It is concluded that $f^*(z)$ possesses the constant dilatation

$$K = \frac{1}{4}\left[\left(\frac{\alpha}{a}\right)^2 - \left(\frac{\beta}{b}\right)^2\right]$$

throughout $\operatorname{Int}\Delta_z$ with possible exception of a discrete subset, where ω_{f^*} vanishes. We shall have to mention about the behaviour of $f^*(z)$ in a neighbourhood of possible zeros of the differential $\omega_{f^*} = p_{f^*}\overline{q_{f^*}}dz^2$. Let z_0 be the zero point of ω_{f^*} of degree $m \geq 1$. Then the $2m+1$ members of $\mathcal{T}$ ends at this point z_0 with the equal angles $\mathrm{A}_n = 2\pi/(2m+1) \quad (n = 1, ..2m+1)$. Similarly, the $2m+1$ members of $\mathcal{O}$ ends at the same point z_0 forming the equal angles $\mathrm{B}_n = 2\pi/(2m+1) \quad (n = 1, ..., 2m+1)$, where every member of A_n is bisected by a certain member of $\mathrm{B}_{n'}$ and conversely every member of B_n is bisected by a certain member of $\mathrm{A}_{n'} \quad (n, n' = 1, ...2m+1)$.

The harmonic homeomorphism $f^*(z)$ put the trajectories $\mathcal{T}$ (resp.the orthogonal trajectories $\mathcal{O}$) to the orthogonal trajectories $\mathcal{O}'$ (resp. trajectories $\mathcal{T}'$) of f^{*-1} into correspondence. Therefore the mapping $f^*(z)$ is conformal at every posssible zero of ω_{f^*}. We may conclude that $f^*(z)$ is a Teichmüller map: Thus we have proved

Proposition 6. $\Gamma_E \subseteq \Gamma_Q$.

From Propositions 1 and 5 follows

Theorem 5. $\Gamma_Q = \Gamma_E$.

Remark 2. The above Theorem 5 is the disk version of Beurling-Ahlfors quasiconformal extension theorem (cf. [1]), which provides, together with Lemma 1 a very simple sufficient condition for quasiconformal extensibility of the peripheral homeomorphism γ (cf. [2]).

Now note that the above argumentations deriving Theorem 5 is based upon the Gerstenhaber-Rauch variation

$$z \mapsto \tilde{z} = z + \varepsilon h(z) \qquad (h(z) \equiv 0 \text{ on } \partial\Delta_z). \tag{4}$$

We are allowed to choose a distortion other than (3) using $h(z)$ vanishing everywhere on *an upper half-disk* surrounded partly by a characteristic cross-cut. The stationary funciton in this case gives also a quasiocnformal homeomorphism of $\operatorname{Int}\Delta_z$ onto $\operatorname{Int}\Delta_w$ with constant dilatations K and K' on the upper and lower half disk respectively, where it holds however that $K \neq K'$.

Theorem 6. *For a given circumferential homeomorphism γ, the extremal quasiconformal homeomorphism of the disk* $\operatorname{Int}\Delta_z$ *onto another such* $\operatorname{Int}\Delta_w$ *is not unique*

(cf. [3]).

References

[1] A. Beurling and L. V. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta Math. 96 (1956) pp.125-142.

[2] Th. M. Rassias and K. Shibata, *Teichmüller extensibility of circle homeomorphisms*, Complex Variables 47(2002), pp.679-706.

[3] K. Shibata, *Stationary functions for Dirichlet's energy functional*, preprint.

[4] K. Strebel, *Zur Frage der Eindeutigkeit extremaler quasikonformer Abbildungen des Einheitskreises* I, Comment. Math. Helv. 36 (1962), pp.306-323.

[5] ———, *Zur Frage der Eindeutigkeit extremaler quasikonformer Abbildungen des Einheitskreises* II, Comment. Math. Helv. 39 (1964), pp.77-89

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