NEVANLINNA, SIEGEL, AND CREMER

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1. Introduction

Let f be a rational function of degree ≥ 2. The *Fatou set* $F(f)$ is defined by the set of all points of $\hat{\mathbb{C}}$ at which $\{f^n\}_{n\in\mathbb{N}}$ is normal in the Montel sense, and the *Julia set* $J(f)$ is defined by the complement of $F(f)$ in $\hat{\mathbb{C}}$. Both $F(f)$ and $J(f)$ are *completely invariant*, that is, their image and preimage by f equal themselves. $F(f)$ is open by definition, so $J(f)$ is closed. Furthermore $J(f)$ is non-empty and perfect.

We call a connected component of $F(f)$ a *Fatou component*. Every Fatou component is mapped to a Fatou one properly by f. A Fatou component D is *cyclic* if for some $n \in \mathbb{N}$, $f^n(D) = D$. Then the least such n is called the *period*, and $g := g_D := f^n|D$ is the *first return map* on D. On the other hand, a Fatou component is *preperiodic* if it is not cyclic but for some $n \in \mathbb{N}$, $f^n(D)$ is cyclic.

The classification of cyclic Fatou components is known: (g, D) is an *attractive basin* if $\{g^n\}_{n=1}^{\infty}$ converges to a point in D locally uniformly on D. The *parabolic basin* is similar, but $\{g^n\}_{n=1}^{\infty}$ converges to a point in the boundary of D locally uniformly on D. In both cases, g is a proper selfmap of D of degree ≥ 2. If g is a univalent selfmap of D, (g, D) is called a *singular domain* since people before doubted whether this case actually occurred. In this case, (g, D) is conformally conjugate to an irrational rotation on either a disk or an annulus, and called a *Siegel disk* or an *Herman ring* respectively. Hence the singular domain is also called the *rotation* one.

We focus on the cycles of points or *circles* associated to rotation domains.

Definition 1.1 (irrationally indifferent cycle of points or circles)**.** A point z_0 in $\hat{\mathbb{C}}$ is *periodic* if for some $p \in \mathbb{N}$, $f^p(z_0) = z_0$. The least such p is the *period* of z_0, $\{f^n(z_0)\}_{n=1}^{p}$ is a *cycle* of f, and $\lambda := (f^p)'(z_0)$ is the *multiplier* of it. This cycle of *points* is *irrationally indifferent* if $\lambda = e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{R} - \mathbb{Q}$.

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A topological circle $S \subset \hat{\mathbb{C}}$ is *periodic* if for some $p \in \mathbb{N}$, $f^p(S) = S$ *and* for every $j \in \mathbb{N}$, $f|f^{j-1}(S)$ is a homeomorphism from $f^{j-1}(S)$ to $f^j(S)$. The least such p is the *period* of S. The *multiplier* of the cycle $\{f^n(S)\}_{n=1}^p$ of *circles* is defined by $e^{2\pi i\alpha}$, where α is the *rotation number* of such a S^1-homeomorphism ϕ that is topologically conjugate to $f^p|S : S \to S$. Hence α equals the (uniform) limit $\lim_{n\to\infty}(\tilde{\phi} - id_{\mathbb{R}})/n$, modulo $\mathbb{Z}$, for a lift $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$ of ϕ. This cycle of *circles* is *irrationally indifferent* if α is irrational.

It is known that if irrationally indifferent cycles of points or circles intersect $F(f)$, then they are contained in some rotation domains which are Fatou components.

Definition 1.2 (Siegel and Cremer cycles)**.** An irrationally indifferent cycle of points or circles is a *Siegel cycles* if it is contained in $F(f)$. Otherwise it is a *Cremer cycle.*

We study an unsolved problem with a *long* history: *Given an irrationally indifferent cycle of points or circles, how can we judge whether it is contained in the Fatou set or not?*

The following answers the problem in one direction, that is, tells us when it is Siegel.

Theorem 1.1 (Siegel[5] (1942) - Rüssmann - Brjuno[1] (1972) - Yoccoz[6] (1996))**.** *Let* $\lambda = e^{2\pi i\alpha}$ $(\alpha \in \mathbb{R} - \mathbb{Q})$, $f(z) = \lambda z + \cdots$ *be an analytic germ at the origin, and* $\{p_n/q_n\}_{n=1}^\infty$ *be the sequence of irreducible approximating fractions of* α *derived from its continued fraction expansion.*

If α *satisfies one of Diophantine conditions*:

$$\text{(Si)} \qquad \sup_{n\geq 0} \frac{\log q_{n+1}}{\log q_n} < \infty,$$

or more weakly,

$$\text{(Rü)} \qquad \sum_{n>0} \frac{\log q_{n+1} \log\log q_{n+1}}{q_n} < \infty,$$

or

$$\text{(Br)} \qquad \sum_{n>0} \frac{\log q_{n+1}}{q_n} < \infty,$$

then f *is analytically linearizable at the origin, that is, the Schröder equation*

$$(1) \qquad h \circ f = R_\lambda \circ h,$$

where $R_\lambda(z) = \lambda z$ *is the linear term of* f*, holds for some analytic local coordinate* $h = z + \cdots$ *around the origin.*

In the reverse direction,

Theorem 1.2 (Yoccoz[6] (1996), Okuyama[4] (2001))**.** *Let P be an quadratic polynomial. An irrationally indifferent cycles of points* (*of arbitrary period*) *of P is Cremer if its multiplier does not satisfy* (*Br*)*.*

Only for quadratic polynomials, the complete answer is known. The classical Cremer Theorem also tell us when an irrationally indifferent cycle is Cremer:

Theorem 1.3 (Cremer[2] (1932))**.** *Let f be a rational function of degree $d \geq 2$, and $\mathcal{O}$ be an irrationally indifferent cycle of points of period p and of multiplier λ.*

$\mathcal{O}$ is Cremer if λ satisfies

$$\text{(Cr)} \qquad \limsup_{n\to\infty} \frac{1}{d^{pn}} \log \frac{1}{|\lambda^n - 1|} = \infty.$$

Before obtaining the complete answer in general case, we would also like to study more fundamental question naturally arising: *How can we notice such complicate conditions as* (*Si*)*,* (*Rü*)*,* (*Br*) *and* (*Cr*)*? What on earth are they?*

In the paper [3], we answer this fundamental question at least for the condition (Ok) below, which not only greatly improves (Cr) but also extends it to irrationally indifferent cycles of circles. We reveal that the left hand side of (Ok) or (Cr) is in fact one of characteristic constants in the *Nevanlinna theory*.

Main Theorem 1 (Non-linearizability Criteria)**.** *Let f be a rational function of degree $d \geq 2$, and $\mathcal{O}$ be an irrationally indifferent cycle of points or circles of period p and of multiplier λ. If*

$$\text{(Ok)} \qquad \limsup_{n\to\infty} \frac{1}{d^{pn}} \log \frac{1}{|\lambda^n - 1|} > 0,$$

then $\mathcal{O}$ is Cremer.

References

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