Conformal imbeddings of domains

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Oikawa's problem I

Welding of polygons and the type of Riemann surfaces were considered by Nevanlinna, Oikawa and others. We are concerned with the relation of weldings and the moduli of Riemann surfaces. Oikawa studied this subject and got some results which he didn't publish. We follow him. A square in the complex plane can be conformally welded into various ring domains by a specific kind of identification of a pair of opposite sides. We consider the range of these moduli. Oikawa gave an estimate for the range of these moduli and asked whether it is the best possible or not.

Let

$$Q = \{x + iy : 0 < x < 1,\ 0 < y < 1\},$$

$$L_+ = \{x + iy : 0 < x < 1,\ y = 1\},$$

$$L_- = \{x + iy : 0 < x < 1,\ y = 0\},$$

$$\phi_0(x + i) = x\ (0 < x < 1),$$

$\phi : L_+ \longrightarrow L_-$ be a homeomorphism such that $\phi \circ \phi_0^{-1}(x)$ is strictly increasing,

$G:$ a ring domain,

$C:$ a Jordan curve in G joining two boundary component of G,

$f : Q \cup L_+ \cup L_- \to G$ a continuous mapping.

(G, C, f) is a conformal welding obtained by ϕ

if f is conformal in Q, $f \circ \phi = f$ on L_+, and $f(L_+) = f(L_-) = C$.

We call ϕ a welding function. A conformal welding by ϕ is unique, if for $\forall(G_i, C_i, f_i)_{i=1,2}$ of conformal weldings obtained by ϕ, $f_2 \circ f_1^{-1}$ is a conformal mapping from G_1 to G_2. Let $M(G) = \log(R_2/R_1)$ be the modulus of G, where G is conformally equivalent to $\{z : R_1 < |z| < R_2\}$. For a welding function ϕ, set
$M_\phi = \{M(G)\ :\ (G, C, f)$ is a conformal welding by $\phi\}$.

Oikawa [5] proved the following three theorems.

Theorem A

$M_\phi = \{2\pi\}$ if and only if $\phi = \phi_0$.

Theorem B

Let Φ be the set of welding functions ϕ, then $\bigcup_{\phi\in\Phi} M_\phi = (0, 2\pi]$.

Theorem C

$$M_\phi \subset [2\pi \int_0^1 \frac{min(\phi'(x), 1)}{1 + (\phi(x) - x)^2} dx, 2\pi].$$

Further Oikawa [5] presented the following problem.

Find a welding function ϕ satisfying

$$M_\phi \supset (2\pi \int_0^1 \frac{min(\phi'(x), 1)}{1 + (\phi(x) - x)^2} dx, 2\pi).$$

We obtain the following.

Theorem

There are a welding function ϕ and $\epsilon > 0$ such that $M_\phi \supset (0, \epsilon)$.

Theorem

For any $\phi \neq \phi_0$, there is an $m < 2\pi$ such that $M_\phi \subset (0, m]$.

Lemma

Let A be an annulus and K_n be constructed as above. For any $\epsilon > 0$, there exists a q.c. mapping h on A which is conformal on $A - K_n$ and satisfies $M(h(A)) < \epsilon$.

Oikawa's problem II

Let

$$A = \{z; a < |z| < 1\} \text{ and } B = \{z; 1 < |z| < b\},$$

φ be a continuous function on $\mathbf{R}$ such that

$$\varphi(\theta_1) < \varphi(\theta_2) \text{ if } \theta_1 < \theta_2,\ \varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi.$$

We call φ a conformal welding function if
$\exists g : A \cup B \to A(\varphi, g) - \gamma$ conformal mapping s.t. $\lim_{A\ni z\to e^{i\theta}} g(z) = \lim_{B\ni z\to e^{i\varphi(\theta)}} g(z)$,
where $A(\varphi, g) = \{w; 1 \leq |w| \leq e^{M(\varphi,g)}\}$, γ is a Jordan closed curve in $A(\varphi, g)$.
We call g a mapping giving (conformal) structure from φ. Set

$$V(\varphi) = \bigcup\{M(\varphi, g); g \text{ is a mapping giving a structure from } \varphi\}.$$

About this set, K. Oikawa asked the following question.

Is there a conformal welding function φ such that $V(\varphi)$ is a point but it has different conformal structures.

Suppose that g and g' give different structure from φ. Then $\exists \tilde{F}$: homeomorphism on $\mathbf{C}$ s.t.

$\tilde{F}$ is quasiconformal on $\mathbf{C} - \gamma$, $\tilde{F} = g' \circ g^{-1}$ on $A(\varphi, g)$.

Further, $\exists h$: quasiconformal mapping on $\mathbf{C}$ s.t. $f = h \circ \tilde{F}$ is conformal on $\mathbf{C} - \gamma$.

Let G_1 be exterior of γ, G_2 inside of γ, U^* exterior of unit disk, U unit disk,

$$\varphi_1(z) = \sum_{n=-1}^{\infty} a_n z^{-n} : U^* \to G_1, \text{ conformal},$$

$$\varphi_2(z) = \sum_{n=0}^{\infty} b_n z^n : U \to G_2, \text{ conformal},$$

$$G_R \ (R > 1) : \text{ bounded region enclosed by } \varphi_1(|z| = R).$$

Then the area

$$|G_R| = \pi\{|a_{-1}|^2 R^2 - \sum_{n=1}^{\infty} n \frac{|a_n|^2}{R^{2n}}\}, \ |\overline{G}_1| = \lim_{R \to 1+} |G_R| = \pi\{|a_{-1}|^2 - \sum_{n=1}^{\infty} n|a_n|^2\}.$$

For G_r $(r < 1)$: bounded region enclosed by $\varphi_2(|z| = r)$,

$$|\underline{G}_1| = \lim_{r \to 1-} |G_r| = \pi \sum_{n=1}^{\infty} n|b_n|^2.$$

If the area of γ vanishes, i.e. $|\overline{G}_1| = |\underline{G}_1|$. Hence

$$|a_{-1}|^2 = \sum_{n=1}^{\infty} n(|a_n|^2 + |b_n|^2).$$

Similary denote

$$\psi_1(z) = \sum_{n=-1}^{\infty} A_n z^{-n} = f \circ \varphi_1, \quad \psi_2(z) = \sum_{n=0}^{\infty} B_n z^n = f \circ \varphi_2,$$

$$\Omega_R \ (R > 1) : \text{ bounded region enclosed by } \psi_1(|z| = R),$$

$$\Omega_r \ (r < 1) : \text{ bounded region enclosed by } \psi_2(|z| = r).$$

We have

$$|\overline{\Omega}_1| = \lim_{R \to 1+} |\Omega_R| = \pi\{|A_{-1}|^2 - \sum_{n=1}^{\infty} n|A_n|^2\},$$

$$|\underline{\Omega}_1| = \lim_{r \to 1-} |\Omega_r| = \pi \sum_{n=1}^{\infty} n|B_n|^2.$$

When the area of $f(\gamma)$ vanishes, we have also

$$|A_{-1}|^2 = \sum_{n=1}^{\infty} n(|A_n|^2 + |B_n|^2).$$

Let a point $t \in \mathbf{C}$ be fixed. Set $g(\zeta) = f(\zeta) + t\zeta$,

$$Q = \sum_{n=1}^{\infty} nB_n\overline{b_n} + \sum_{n=-1}^{\infty} nA_n\overline{a_n}.$$

Then

$$(dg, dg)_{G_R-\gamma} = 2\pi\{2\Re\,\bar{t}Q - \sum_{n=-1}^{\infty} n|A_n + ta_n|^2 R^{-2n}\}.$$

We remark the following.

Lemma

$$Q = \sum_{n=1}^{\infty} nB_n\overline{b_n} + \sum_{n=-1}^{\infty} nA_n\overline{a_n} \neq 0.$$

Lemma

$$|\Omega_R| - \frac{1}{2}(dg, dg)_{G_R-\gamma} = -2\pi\Re\,\bar{t}Q.$$

Let

$$S(f) = \{\frac{f(\zeta_1) - f(\zeta_2)}{\zeta_1 - \zeta_2} : (\zeta_1, \zeta_2) \in \mathbf{C} \times \mathbf{C} - \{(\zeta, \zeta)\}_{\zeta\in\mathbf{C}}\}.$$

We remark the following.

Theorem

$$\{t : \Re\,\bar{t}Q < 0\} \subset S(f).$$

If $-t$ does not belong to $S(f)$, $f(\zeta_1)+t\zeta_1 \neq f(\zeta_2)+t\zeta_2$ for every pair $(\zeta_1, \zeta_2), \zeta_1 \neq \zeta_2$. Then $g(\zeta) = f(\zeta) + t\zeta$ is univalent.

When $-t \in \mathbf{C} - S(f)$ and $\Re\,\bar{t}Q < 0$, g is univalent and the image area of γ by g is positive. Therefore

Theorem *If $(\mathbf{C} - S(f))\bigcap\{t; \Re\,\bar{t}Q > 0\} \neq \emptyset$, there exists a homeomorphism g on $\mathbf{C}$ such that g is conformal on $\mathbf{C} - \gamma$ and area of $g(\gamma)$ is positive. $V(\varphi)$ is not a point.*

Riemann surface and Teichmüller space

Let R be a finite compact bordered Riemann surface of genus p with m boundary components, $\hat{R}$ be the doubled Riemann surface of R, this genus is $2p + m - 1$,

i be an anticonformal mapping (involution) on $\hat{R}$ $i \circ i = identity$ and $R \cup i(R) = \hat{R}$.

Let

$$T(R_0) = \{(R, g); {}^{\exists}g : R_0 \to R \;\; q.c.\}/\sim .$$

where R is a finite compact bordered Riemann surface,

$$(R_1, g_1) \sim (R_2, g_2) \Longleftrightarrow {}^{\exists}h : R_1 \to R_2 \ \ conformal \text{ s.t. } g_2^{-1} \circ h \circ g_1 \simeq identity : homotopic.$$

$T(R_0)$ is reduced Teichmüller space of R_0 and $6p-6+3m$ (except for $p=0,\ m = 0,\ 1,\ 2$) dimensional real analytic manifold.

Embedable Riemann surface

For $R_i = (R_i, g_i) \in T(R_0)$, set

$$T(R_0; R_i) = \{R_j = (R_j, g_j) \in T(R_0); {}^{\exists}h_j : R_i \to R_j : conformally\ into$$

$$\text{s.t. } g_j^{-1} \circ f_j \circ g_i \simeq identity : homotopic\}.$$

Remark.

f_j is conformal on ∂R_i, $h_j(\partial R_i)$: analytic curves on R_j, $m(R_j - f_j(R_i)) > 0$.

Quasiconformal deformation on $R_j - h_j(R_i)$ covers a neighborhood V of R_j in $T(R_0)$. Hence $V \subset T(R_0; R_i)$, $T(R_0; R_i)$ is open.

For $(R_j, g_j) \in T(R_0; R_i)$, set

$$CE(R_i, R_j) = \{f; f : R_i^\circ \to R_j^\circ : conformally\ into \text{ s.t. } g_j^{-1} \circ f \circ g_i \simeq identity : homotopic\},$$

where R_k° denotes the interior of R_k.

Optimal conformal embedding

Let R'_j be a subdomain of R_j° s.t. every component of $R_j^\circ - R'_j$ is doubly connected,

$$Z(R_j^\circ, R'_j) = \{\gamma \subset R_j^\circ - R_j;$$

$$\text{rectifiable closed Jordan curves } \gamma \text{ divides every component of } R_j^\circ - R_j\},$$

$\lambda(Z(R_j^\circ, R'_j))$ be the extremal length of $Z(R_j^\circ, R'_j)$ i.e.

$$\lambda(Z(R_j^\circ, R'_j)) = sup_\rho\{\frac{1}{A(\rho)}; \rho \text{ is a Borel measurable conformal density}$$

$$\text{s.t. } inf_{\gamma \in Z(R_j^\circ, R'_j)}\{\int_\gamma \rho(z)|dz|\} \geq 1\}, \text{ where } A(\rho) = \int\int_{R_j} \rho^2(x+iy)dxdy.$$

For $f \in CE(R_i, R_j)$, $\lambda(Z(R_j^\circ, f(R_i^\circ))) = \lambda(Z({R'_j}^\circ, h_j \circ f \circ h_i^{-1}({R'_i}^\circ)))$. Set

$$B(R_i, R_j) = inf\{\lambda(Z(R_j^\circ, f(R_i^\circ))); f \in CE(R_i, R_j)\},$$

where $B(R_i, R_j) = \infty$ if $CE(R_i, R_j)$ is empty.

Theorem. *Suppose* $B(R_i, R_j) < \infty$.

${}^{\exists}f_{ij} \in CE(R_i, R_j)$ *s.t.* $\lambda(Z(R_j^\circ, f_{ij}(R_i^\circ))) = B(R_i, R_j)$. *The boundary of* $f_{ij}(R_i^\circ)$

consists of trajectories of a quadratic holomorphic differential φ_j on R_j; hence the boundary is analytic.

$f_{ij}(R_i)$: optimal conformal embedding from R_i to R_j.

H be a harmonic function on $R_j - f_{ij}(R_i)$ s.t. $H = \begin{cases} 1 & \textit{on } \partial R_j \\ 0 & \textit{on } \partial f_{ij}(R_i). \end{cases}$

Then

$$(\frac{\partial}{\partial z}H)^2 dz^2 = \varphi_j \text{ on } R_j - f_{ij}(R_i).$$

For a t $(0 < t \leq 1)$

$$R_{jt} = f_{ij}(R_i) \cup \{z \in R_j - f_{ij}(R_i); H(z) \leq t\} \in T(R_0; R_i).$$

R_{jt} and $R_{j1} = R_j$ are arcwise connected in $T(R_0; R_i)$.

Proposition. *Let $6p - 6 + 3m > 0$. Then*
$T(R_0; R_i)$ is a $6p - 6 + 3m$ dimensional real analytic submanifold of $T(R_0)$.

Ishida's example

Let

$$G = \hat{\mathbf{C}} - [-1, 0] \cup [1, 2] \cup [3, \infty],$$

$$G_\epsilon = \hat{\mathbf{C}} - [-1, 0] \cup [1, 2 + \epsilon] \cup [3 - \epsilon . \infty] \ (0 < \epsilon < \frac{1}{2}).$$

There is no subdomain G' of G such that G' is conformal to G_ϵ and $G - G'$ has a positive measure.

Denjoy domain

Let

$$D(a_i, b_i) = \hat{\mathbf{C}} - \cup_{i=1}^{n}[a_i, b_i], \text{where } a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < ...... < a_n < b_n.$$

$D(a_i', b_i')$ is a Denjoy proper subdomain of $D(a_i, b_i)$ if $[a_i, b_i] \subset [a_i', b_i']$ for each $i(1 \leq i \leq n)$. Consider a Re-imbedding f of $D(a_i', b_i')$ into $D(a_i, b_i)$ i.e. f is coformal into $D(a_i, b_i)$ from $D(a_i', b_i')$ s.t. $f(z) \to E_i$ if and only if $z \to [a_i', b_i']$, where E_i is a component of $\hat{\mathbf{C}} - f(D(a_i', b_i'))$ which contains $[a_i, b_i]$.

Theorem *For $n \geq 3$, re-imbedding f, $\hat{\mathbf{C}} - f(D(a_i', b_i'))$ has no interior point.*

Theorem *For $n \geq 3$, if $a_1 = a_1', b_1 = b_1', a_2 = a_2', b_n = b_n'$,re-imbedding f is only identity mapping.*

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Kiitos! Näkemiin!