POTENTIAL THEORETIC CHARACTERIZATIONS OF NONSMOOTH DOMAINS
— A CONVERSE OF THE BOUNDARY HARNACK PRINCIPLE —

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A large number of works have been devoted to the study on potential theory for nonsmooth domains, such as Lipschitz domains, NTA domains, uniform domains, John domains and Hölder domains, since the pioneering works of Carleson [10] and Hunt-Wheeden [15, 16] for Lipschitz domains. Among them, the boundary Harnack principle (abbreviated to BHP) and the identification of the Martin boundary played a central role. See Ancona [4], Dahlberg [11] and Wu [23] for Lipschitz domains, Jerison-Kenig [17] for NTA domains, Bass-Burdzy [8] for Hölder domains, [1] and [3] for uniform domains and uniformly John domains. These studies are motivated to generalize the geometric conditions imposed on a domain to guarantee potential theoretic properties, such as the boundary Harnack principle.

This paper is in the opposite direction. Namely, we shall give some potential theoretic properties which yield geometric properties of the domain. Our conditions will be necessary and sufficient, provided the domain satisfies the capacity density condition. To be more precise, we use the following notation. Throughout the paper we let D be a bounded domain in $\mathbb{R}^n$ with $n \ge 2$ and let $\delta_D(x) = \operatorname{dist}(x, \partial D)$. We write $B(x,r)$ and $S(x,r)$ for the open ball and the sphere of center at x and radius r, respectively. By the symbol A we denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use $A_0, A_1, \ldots,$ to specify them. We shall say that two positive quantities f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $A \ge 1$ such that $A^{-1} f_1 \le f_2 \le A f_1$. The constant A will be called the constant of comparison.

Let us recall the definitions of some nonsmooth domains. We say that D is a *John domain* with John constant $c_J > 0$ and John center $x_0 \in D$ if each $x \in D$ can be joined to x_0 by a rectifiable curve γ such that

$$\delta_D(y) \ge c_J \ell(\gamma(x,y)) \quad \text{for all } y \in \gamma, \tag{1}$$

where $\gamma(x,y)$ and $\ell(\gamma(x,y))$ stand for the subarc of γ connecting x and y and its length, respectively. In general, $0 < c_J < 1$. A John domain may be visualized as a domain satisfying a twisted cone condition. We say that D is a *uniform domain* if there exist constants A and A' such that each pair of points $x, y \in D$ can be joined by a rectifiable curve $\gamma \subset D$ such that $\ell(\gamma) \le A|x-y|$ and

$$\min\{\ell(\gamma(x,z)), \ell(\gamma(z,y))\} \le A\delta_D(z) \quad \text{for all } z \in \gamma. \tag{2}$$

See Gehring-Martio [13] and Väisälä [22]. If the complement of a uniform domain D satisfies the corkscrew condition, then D becomes an NTA domain (Jerison-Kenig [17]). We define the *internal metric* $\rho_D(x,y)$ by

$$\rho_D(x,y) = \inf\{\operatorname{diam}(\gamma) : \ \gamma \text{ is a curve connecting } x \text{ and } y \text{ in } D\}$$

for $x, y \in D$. Here $\operatorname{diam}(\gamma)$ denotes the diameter of γ. Obviously $|x-y| \le \rho_D(x,y)$. We say that D is a *uniformly John domain* if there exists a constant $A > 1$ such that each pair of points $x, y \in D$ can be connected by a curve $\gamma \subset D$ with $\ell(\gamma) \le A\rho_D(x,y)$ and (2). By definition a uniformly John domain is a domain intermediate between a John domain and a uniform domain. The above definition is due to Balogh-Volberg [6, 7]. Bonk-Heinonen-Koskela [9] called a uniformly John domain an *inner uniform* domain and showed an inner uniform domain is a Gromov hyperbolic domain.

Now we consider potential theoretic properties and their connections to the above non-smooth domains.

Definition 1. We say that a domain D enjoys the *uniform BHP* if there exist constants $A_0,\ A_1 > 1$ and $r_0 > 0$ depending only on D with the following property: Let $\xi \in \partial D$ and let $0 < r < r_0$. Suppose u and v are positive harmonic functions on $D \cap B(\xi, A_0 r)$, bounded on $D \cap B(\xi, A_0 r)$ and vanishing on $\partial D \cap B(\xi, A_0 r)$ except for a polar set. Then

$$\frac{u(x)/u(y)}{v(x)/v(y)} \le A_1, \quad \text{whenever } x, y \in D \cap B(\xi, r). \tag{3}$$

In [1, Theorem 1], we proved the following.

Theorem A. *A uniform domain satisfies the uniform BHP.*

We shall give a converse of Theorem A. The definition of a uniform domain is very sensitive about the boundary; if we remove a closed polar set from the domain D, then D may not be a uniform domain, whereas the uniform BHP remains to hold. So, we need some additional assumption on the boundary in order to obtain a converse. The following *capacity density condition* (abbreviated to CDC) is reasonable and widely known. Let U be an open set with Green function G_U. Define the Green capacity $\mathrm{Cap}_U(E)$ for a Borel set $E \subset U$ by

$$\mathrm{Cap}_U(E) = \sup\{\mu(E) : G_U\mu \le 1 \text{ on } U,\ \mu \text{ is a Borel measure supported on } E\}.$$

In the usual way $\mathrm{Cap}_U(E)$ extends to a general set $E \subset U$.

Definition 2. We say that the CDC holds if there exist constants $A > 1$ and $r_0 > 0$ such that

$$\frac{\mathrm{Cap}_{B(\xi,2r)}(B(\xi, r) \setminus D)}{\mathrm{Cap}_{B(\xi,2r)}(B(\xi, r))} \ge \frac{1}{A}, \quad \text{whenever } \xi \in \partial D \text{ and } 0 < r < r_0.$$

Remark 1. If D satisfies the CDC, then D is regular and the assumption of u and v for the BHP becomes that u and v are positive and harmonic on $D \cap B(\xi, A_0 r)$ and continuously vanish on $\partial D \cap B(\xi, A_0 r)$. In the sequel, we assume the CDC and use this simplified assumption.

Ancona [5, Lemma 3] showed that the CDC has an equivalent condition in terms of harmonic measure. We write $\omega(\cdot, E, U)$ for the harmonic measure over an open set U of $E \subset \partial U$. Then the CDC holds if and only if there exist constants $\beta > 0$, $A > 1$ and $r_0 > 0$ such that

$$\omega(x, D \cap S(\xi, r), D \cap B(\xi, r)) \le A\left(\frac{|x - \xi|}{r}\right)^{\beta} \quad \text{for } x \in D \cap B(\xi, r), \tag{4}$$

whenever $\xi \in \partial D$ and $0 < r < r_0$. See also [2] for the connection to the Dirichlet problem. Under the assumption of the CDC, we can characterize a John domain, a uniform domain and a uniformly John domain in terms of potential theoretical properties.

Theorem 1. *Let D satisfy the CDC. Then D is a John domain if and only if there exist constants $\alpha > 0$, $A > 1$ and $r_0 >$ such that*

$$\omega(x, D \cap S(\xi, r), D \cap B(\xi, r)) \geq \frac{1}{A}\left(\frac{\delta_D(x)}{r}\right)^{\alpha} \quad \text{for } x \in D \cap B(\xi, \frac{r}{A}), \tag{5}$$

whenever $\xi \in \partial D$ and $0 < r < r_0$.

The following theorem includes a converse of Theorem A.

Theorem 2. *Let D satisfy the CDC. Then D is a uniform domain if and only if the uniform BHP and* (5) *hold.*

Remark 2. Jerison-Kenig [17] developed a fruitful potential theory on an NTA domain, including the BHP and the CE. An NTA domain can be regarded as a uniform domain with the corkscrew condition of the complement. The requirement of the complement, needed for the construction of a uniform barrier, may be replaced by a more general condition, CDC. Thus the arguments of Jerison-Kenig remain to hold for a uniform domain with CDC. Our Theorem 2 asserts its converse, i.e., if a domain enjoys the CDC, the BHP and (5), then it must be a uniform domain. On the other hand, Theorem A does not rely on a uniform barrier and, as a result, it is valid for a uniform domain without CDC. Theorem A was derived in a spirit of Bass-Burdzy [8] rather than by the method of Jerison-Kenig. Characterization of a uniform domain without CDC remains open.

Remark 3. Uniform domains are known to be *nice* domains of analysis appearing in many contexts (Jones [18, 19], Gehring [12], Gotoh [14], Bonk-Heinonen-Koskela [9]), although their boundaries may be very complicated. Our characterization provides another example of this nature. See also Smith-Stegenga [20] and Stegenga-Ullrich [21] for characterizations of a Hölder domain, a domain satisfying the quasihyperbolic metric condition.

A ball with respect to the internal metric becomes a connected component of the intersection of a Euclidean ball and the domain ([3, Lemma 2.2]). So, we arrive at the following version of the uniform BHP, which is a property weaker than the uniform BHP.

Definition 3. We say that a domain D enjoys the *uniform BHP* with respect to the internal metric if there exist constants $A_2,\ A_3 > 1$ and $r_0 > 0$ depending only on D with the following property: Let $\xi \in \partial D$, $0 < r < r_0$, U a connected component of $D \cap B(\xi, r)$ and V a connected component of $D \cap B(\xi, A_2 r)$ including U. Suppose u and v are positive harmonic functions on V vanishing on $\partial D \cap \partial V$. Then

$$\frac{u(x)/u(y)}{v(x)/v(y)} \leq A_3, \quad \text{whenever } x, y \in U.$$

In [3, Theorem 3.1], we proved the following.

Theorem B. *A uniformly John domain satisfies the uniform BHP with respect to the internal metric.*

The following characterization includes a converse of Theorem B.

Theorem 3. *Let D satisfy the CDC. Then D is a uniformly John domain if and only if the uniform BHP with respect to the internal metric and* (5) *hold.*

It is known that a finitely connected planar domain without singleton boundary components satisfies the CDC. Hence we have the following corollaries.

Corollary 1. *Let D be a bounded finitely connected planar domain. Then D is a John domain if and only if* (5) *holds.*

Corollary 2. *Let D be a bounded finitely connected planar domain. Then D is a uniform domain if and only if the uniform BHP and* (5) *hold.*

Corollary 3. *Let D be a bounded finitely connected planar domain. Then D is a uniformly John domain if and only if the uniform BHP with respect to the internal metric and* (5) *hold.*

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